BLOCKS AND MODULES FOR WHITTAKER PAIRS

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ABSTRACT. Inspired by recent activities on Whittaker modules over various (Lie) algebras we describe some general framework for the study of Lie algebra modules locally finite over a subalgebra. As a special case we obtain a very general setup for the study of Whittaker modules, which includes, in particular, Lie algebras with triangular decomposition and simple Lie algebras of Cartan type. We describe some basic properties of Whittaker modules, including a block decomposition of the category of Whittaker modules and certain properties of simple Whittaker modules under some rather mild assumptions. We establish a connection between our general setup and the general setup of Harish-Chandra subalgebras in the sense of Drozd, Futorny and Ovsienko. For Lie algebras with triangular decomposition we construct a family of simple Whittaker modules (roughly depending on the choice of a pair of weights in the dual of the Cartan subalgebra), describe their annihilators and formulate several classification conjectures. In particular, we construct some new simple Whittaker modules for the Virasoro algebra. Finally, we construct a series of simple Whittaker modules for the Lie algebra of derivations of the polynomial algebra, and consider several finite dimensional examples, where we study the category of Whittaker modules over solvable Lie algebras and their relation to Koszul algebras.

1. INTRODUCTION

The original motivation for this paper stems from the recent activities on Whittaker modules for some infinite dimensional (Lie) algebras, which resulted in the papers [On1, On2, Ta, BO, Ch, OW, TZ, LW, Wa, LWZ, WZ]. Whittaker modules for the Lie algebra $\mathfrak{sl}_2$ appear first in the work [AP] of Arnal and Pinczon. For all simple finite dimensional complex Lie algebras they were constructed by Kostant in [Ko]. As in the "classical" highest weight representation theory of Lie algebras, Whittaker modules are associated to a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. However, in contrast to this "classical" theory, Whittaker modules are not weight modules, which means that the action of nonzero elements from the Cartan subalgebra $\mathfrak{h}$ on Whittaker modules is (usually) not diagonalizable. For simple finite dimensional complex Lie algebras Whittaker modules subsequently appeared in connection

to parabolic induction and related generalizations of the BGG category $\mathcal{O}$, see [MD1, MD2, Ba, MS1, MS2, KM].

In the recent papers mentioned above Whittaker modules are constructed and studied for some deformations of $\mathfrak{sl}_2$, in particular, for generalized Weyl algebras; and for several infinite dimensional Lie algebras, in particular, for the Virasoro algebra and its various generalizations. At the same time, there are many other Lie algebras for which one can naturally define Whittaker modules, notably, affine Kac-Moody algebras, Witt algebras, Lie algebras of Cartan type. Our main motivation for the present paper was an attempt to understand the general picture of "Whittaker type modules" for Lie algebras. We managed to find only one attempt to define some general setup in [Wa], however, the setup of that paper is still rather restrictive and mostly directed to some generalizations of the Virasoro algebra (for example, it does not cover the general case of affine Kac-Moody algebras and even simple finite dimensional Lie algebras).

Comparison of the methods used in the papers listed above suggests the following common feature of the situation: we have some Lie algebra $\mathfrak{g}$ (possibly infinite dimensional) and some subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ (also possibly infinite dimensional, and usually a kind of "nilpotent"). Whittaker modules associated to the pair $\mathfrak{n} \subset \mathfrak{g}$ are $\mathfrak{g}$-modules, generated by a one-dimensional $\mathfrak{n}$-invariant subspace. Moreover, the pair $\mathfrak{n} \subset \mathfrak{g}$ is chosen such that the action of $\mathfrak{n}$ on any Whittaker $\mathfrak{g}$-module is locally finite. Generalizing this situation it is natural to start with a pair $\mathfrak{n} \subset \mathfrak{g}$ of Lie algebras and look for $\mathfrak{g}$-modules, the action of $\mathfrak{n}$ on which is locally finite. This is our general setup in the present paper. Apart from Whittaker modules this setup also includes classical weight modules (in the case when the subalgebra $\mathfrak{n}$ is chosen to be a Cartan subalgebra of $\mathfrak{g}$), and classical Harish-Chandra and generalized Harish-Chandra modules, see e.g. [Vo, PZ]. However, we will see that some of the results, repeatedly appearing in the papers devoted to Whittaker modules in special cases, may be proved already in such a general setup, or under some mild, but still general restrictions.

Let us now briefly describe the content of the paper. In Section 2 we describe the general setup to study $\mathfrak{g}$-modules locally finite over a subalgebra $\mathfrak{n}$. We show that under some restrictions such modules form a Serre subcategory in the category of all modules; and that examples of such modules can be obtained using the usual induction functor provided that the adjoint module $\mathfrak{g}/\mathfrak{n}$ is itself locally finite over $\mathfrak{n}$. We also describe a block decomposition for the category of such modules. In Section 3 we study, as a special case, a general setup for Whittaker modules, where we assume that the subalgebra $\mathfrak{n}$ is quasi-nilpotent and acts locally nilpotent on the adjoint module $\mathfrak{g}/\mathfrak{n}$. We define Whittaker modules in this situation and describe their basic properties. Using the general block decomposition theorem, obtained in Section 2, we

prove that blocks of the category of Whittaker modules trivialize in the sense that they bijectively correspond to characters of $\mathfrak{n}$. We also show that in the case when $\mathfrak{n}$ is finite dimensional, the algebra $U(\mathfrak{n})$ is a Harish-Chandra subalgebra of $U(\mathfrak{g})$ in the sense of Drozd, Futorny and Ovsienko ([DFO]). In Section 4 we study simple Whittaker modules and their connection to one-dimensional $\mathfrak{n}$-modules. For Lie algebras with triangular decompositions we construct a class of simple Whittaker modules as submodules in completions of usual Verma modules. We show that such simple Whittaker modules inherit the annihilator from the corresponding Verma module. We also formulate some conjectures on the structure of simple and induced Whittaker modules. In Section 5 we consider a concrete example of the Lie algebra $\mathfrak{w}_n$ of derivations of a polynomial algebra. We construct two Whittaker pairs associated with this algebra, one with a finite dimensional nilpotent subalgebra and the other, a kind of the "opposite one", with an infinite dimensional quasi-nilpotent subalgebra. We briefly review the theory of highest weight and lowest weight modules and use it to construct a series of simple Whittaker modules for $\mathfrak{w}_n$. Generalizing some of the arguments we prove a general theorem on the existence of simple Whittaker modules with a one-dimensional space of Whittaker vectors. Finally, we complete the paper with several examples for solvable finite dimensional algebras in Section 6. Here the description of simple modules seems to be relatively easy, so the most interesting question is about the category of Whittaker modules (say, those of finite length). We believe that blocks of this category are in some cases described by Koszul algebras.

2. General setup

2.1. **Categories of locally finite dimensional modules.** In this paper we will work over the field $\mathbb{C}$ of complex numbers, so all unspecified vector spaces, homomorphisms and tensor products are taken over $\mathbb{C}$. For a Lie algebra $\mathfrak{a}$ we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. For simplicity we assume that all vector spaces (in particular all algebras) have at most countable dimension. For an abelian category $\mathfrak{A}$ of modules over some algebra we denote by $\overline{\mathfrak{A}}$ the full subcategory of $\mathfrak{A}$, consisting of all modules of finite length.

Let $\mathfrak{g}$ be a nonzero complex Lie algebra (possibly infinite dimensional) and $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$. If M is a $\mathfrak{g}$-module, then we say

that the action of $\mathfrak{n}$ on M is *locally finite* provided that $U(\mathfrak{n})v$ is finite dimensional for all $v \in M$. Let $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ denote the full subcategory of the category $\mathfrak{g}$-Mod of all $\mathfrak{g}$-modules, which consists of all $\mathfrak{g}$-modules, the action of $\mathfrak{n}$ on which is locally finite. Obviously, $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ is an abelian subcategory of $\mathfrak{g}$-Mod with usual kernels and cokernels.

Proposition 1. *If $\mathfrak{n}$ is finite dimensional, then $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ is a Serre subcategory of $\mathfrak{g}$-Mod.*

Proof. The nontrivial part of this claim is to show that for any short exact sequence

$$0 \to X \to Y \to Z \to 0$$

in $\mathfrak{g}$-Mod we have that $X, Z \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ implies $Y \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$. Let $v \in Y$. Then $U(\mathfrak{n})v/(U(\mathfrak{n})v \cap X)$ is finite dimensional as $Z \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$. At the same time, the set

$$I = \{u \in U(\mathfrak{n}) : uv \in X\}$$

is a left ideal of $\mathfrak{n}$. As $\mathfrak{n}$ is finite dimensional, the algebra $U(\mathfrak{n})$ is left noetherian ([Di, 2.3.8]) and hence I is finitely generated, say by some elements $u_1, \ldots, u_k$. This means that $U(\mathfrak{n})v \cap X$ coincides with the sum of the subspaces $U(\mathfrak{n})u_i v$, $i = 1, \ldots, k$, each of which is finite dimensional as $X \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$. Hence $U(\mathfrak{n})v \cap X$ and thus also $U(\mathfrak{n})v$ are finite dimensional as well and the claim follows. $\square$

From the definition of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ we have that a $\mathfrak{g}$-module M belongs to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ if and only if there is a filtration,

$$(1) \qquad 0 = M_0 \subset M_1 \subset M_2 \subset \ldots, \qquad M = \bigcup_{i=0}^{\infty} M_i,$$

of M by finite dimensional $\mathfrak{n}$-invariant subspaces. If X is a finite dimensional $\mathfrak{n}$-module and L is a simple finite dimensional $\mathfrak{n}$-module, we denote by $[X : L]$ the multiplicity of L in X (which is obviously well-defined). For M and L as above we set

$$[M : L] := \sup_{i \geq 0}\{[M_i : L]\} \in \{\infty, 0, 1, 2 \ldots\},$$

and it is easy to see that $[M : L]$ does not depend on the choice of the filtration (1). Using this notation we define the following full subcategories of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$:

- the subcategory $\mathfrak{Q}_{\mathfrak{n}}^{\mathfrak{g}}$, consisting of all M which are semisimple over $\mathfrak{n}$;
- the subcategory $\mathfrak{G}_{\mathfrak{n}}^{\mathfrak{g}}$, consisting of all M such that $[M : L] < \infty$ for all L;
- the subcategory $\mathfrak{H}_{\mathfrak{n}}^{\mathfrak{g}}$ of *Harish-Chandra* modules, consisting of all M which are semisimple over $\mathfrak{n}$ and such that $[M : L] < \infty$ for all L.

If $\mathfrak{C}$ is any of the categories $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$, $\mathfrak{Q}_{\mathfrak{n}}^{\mathfrak{g}}$, $\mathfrak{G}_{\mathfrak{n}}^{\mathfrak{g}}$, $\mathfrak{H}_{\mathfrak{n}}^{\mathfrak{g}}$, then it is natural to ask what are simple objects of $\mathfrak{C}$. One might expect that under some natural assumptions either $\mathfrak{C}$ or $\overline{\mathfrak{C}}$ has a block decomposition with possibly finitely many simple objects in each block. In the latter case it is known that the category $\overline{\mathfrak{C}}$ can be described as the category of finite dimensional modules over some (completed) algebra ([Ga]), and so it is natural to ask what this algebra is.

Example 2. Let $\mathfrak{g}$ be a semisimple finite dimensional complex Lie algebra and $\mathfrak{n} = \mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then the category $\mathfrak{Q}_{\mathfrak{h}}^{\mathfrak{g}}$ is the category of all $\mathfrak{g}$-modules, which are weight with respect to $\mathfrak{h}$. The category $\mathfrak{H}_{\mathfrak{h}}^{\mathfrak{g}}$ is the category of all weight $\mathfrak{g}$-modules with finite dimensional weight spaces. All simple objects in $\mathfrak{H}_{\mathfrak{h}}^{\mathfrak{g}}$ are classified ([Ma]) and it is known that they coincide with simple objects in the category $\mathfrak{G}_{\mathfrak{h}}^{\mathfrak{g}}$.

Example 3. If $\mathfrak{g}$ is the Lie algebra of an algebraic group and $\mathfrak{n}$ is a symmetrizing Lie subalgebra of $\mathfrak{g}$, then $\mathfrak{H}_{\mathfrak{h}}^{\mathfrak{g}}$ consists of usual *Harish-Chandra modules* (e.g. in the sense of [Di, Chapter 9]).

Some important properties of the category $\mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$ are given by the following statements:

Proposition 4. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$. Then the category $\mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$ is stable under the usual tensor product of $\mathfrak{g}$-modules, in particular, the category $\mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$ is a monoidal category.*

Proof. Let $M, N \in \mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$, $v \in M$ and $w \in N$. Then

$$U(\mathfrak{g})(v \otimes w) \subset (U(\mathfrak{g})v) \otimes (U(\mathfrak{g})w),$$

and the latter space is finite dimensional as both $U(\mathfrak{g})v$ and $U(\mathfrak{g})w$ are (because $M, N \in \mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$). The claim follows. $\square$

Proposition 5. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$. Then every nonzero module $M \in \mathfrak{W}_{\mathfrak{h}}^{\mathfrak{g}}$ has a finite dimensional simple $\mathfrak{n}$-submodule.*

Proof. Take any nonzero $v \in M$. Then $U(\mathfrak{n})v$ is a nonzero finite dimensional $\mathfrak{n}$-submodule of M and hence has a well-defined socle (as an $\mathfrak{n}$-module). The claim follows. $\square$

2.2. **The induction functor.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ be a subalgebra of $\mathfrak{g}$. Then we have the usual restriction functor

$$\mathrm{Res}_{\mathfrak{n}}^{\mathfrak{g}} : \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}} \to \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}},$$

which is obviously exact and hence potentially might have a left adjoint and a right adjoint. For the category of all modules, the corresponding left adjoint is the usual induction functor $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}} := U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} -$ and the corresponding right adjoint is the usual coinduction functor $\mathrm{Coind}_{\mathfrak{n}}^{\mathfrak{g}} :=$

$\mathrm{Hom}_{U(\mathfrak{n})}(U(\mathfrak{g}), \_)$. The problem is that these functors do not have to map $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$ to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ in the general case. However, we can state at least the following:

Theorem 6. *If the adjoint $\mathfrak{n}$-module $\mathfrak{g}/\mathfrak{n}$ belongs to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$, then $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}$ maps $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$ to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$. In particular $(\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}, \mathrm{Res}_{\mathfrak{n}}^{\mathfrak{g}})$ is an adjoint pair of functors between $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$ and $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$.*

Proof. We choose a filtration of $\mathfrak{g}/\mathfrak{n}$ of the form (1):

$$0 = X_0 \subset X_1 \subset X_2 \subset \dots, \qquad \bigcup_{i=0}^{\infty} X_i = \mathfrak{g}/\mathfrak{n}.$$

Note that each X_i is a finite dimensional $\mathfrak{n}$-module. Now we choose a special PBW basis of $\mathfrak{g}$ (indexed by a well-ordered at most countable set). If $\mathfrak{n}$ is finite dimensional, we start by choosing a basis of $\mathfrak{n}$, then extend it by some elements which induce a basis of X_1, then extend the result by some elements which induce a basis of X_2/X_1 and so on. If $\mathfrak{n}$ is infinite dimensional, we fix some basis of $\mathfrak{n}$ and then alternate the elements from this basis first with some elements which induce a basis of X_1, then with some elements which induce a basis of X_2/X_1 and so on.

Let $M \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$. Then any element from $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} M$ can be written as a finite sum

$$x = \sum_i u_i \otimes v_i$$

for some $u_i \in U(\mathfrak{g})$ and $v_i \in M$. As this sum is finite, to prove that $\dim U(\mathfrak{n})x < \infty$ it is enough to prove that $\dim U(\mathfrak{n})u_i \otimes v_i < \infty$ for any i. We can write u_i as a finite linear combination of standard monomials in the PBW basis chosen in the previous paragraph. As this linear combination is finite, from our choice of the basis it follows that there exists j such that all basis elements occurring in this linear combination either belong to $\mathfrak{n}$ or descend to elements from X_j. Consider now the element $u(u_i \otimes v_i)$ for some $u \in U(\mathfrak{n})$. Commuting all elements from $\mathfrak{n}$ to the right and moving them through the tensor product we obtain that $u(u_i \otimes v_i)$ lies in the vector space, which can be identified with $X_j \otimes U(\mathfrak{n})v_i$. The latter is a finite dimensional vector space as X_j is finite dimensional and $M \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$. This implies that the action of $U(\mathfrak{n})$ on $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} M$ is locally finite. Therefore $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}$ maps $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$ to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ and the claim of the theorem follows follows from the fact that $(\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}, \mathrm{Res}_{\mathfrak{n}}^{\mathfrak{g}})$ is an adjoint pair of functors between $\mathfrak{n}$-Mod and $\mathfrak{g}$-Mod. $\square$

Corollary 7. *Assume that $\mathfrak{g}/\mathfrak{n} \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$. Then for any finite dimensional $\mathfrak{n}$-module V every submodule of the module $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}V$ has a simple finite dimensional $\mathfrak{n}$-submodule.*

Proof. This follows immediately from Theorem 6 and Proposition 5.
$\square$

Note that, if $\mathfrak{g}$ is finite dimensional, then the condition of Theorem 6 is obviously satisfied for any $\mathfrak{n}$, so one gets that the property to be locally finite dimensional with respect to the action of $U(\mathfrak{n})$ is preserved under induction.

If $\mathfrak{n} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ in the general sense (for example, in the case when $\mathfrak{g}$ is semisimple finite dimensional, or affine Kac-Moody, or the Virasoro algebra, or a Witt algebra, or an algebra of Cartan type), Theorem 6 implies that the $\mathfrak{g}$-module, induced from a weight $\mathfrak{h}$-module, is a generalized weight module. In these cases even a stronger statement is true, namely that the module induced from a weight module is a weight module.

2.3. Block decomposition of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$.

In this subsection we assume that $\mathfrak{g}$ is a Lie algebra and $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$ such that the adjoint $\mathfrak{n}$-module $\mathfrak{g}/\mathfrak{n}$ belongs to $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{n}}$. Let $\mathrm{Irr}_{\mathfrak{n}}^{f}$ denote the set of isomorphism classes of simple finite dimensional $\mathfrak{n}$-modules. If $X \subset \mathrm{Irr}_{\mathfrak{n}}^{f}$ and L is a simple finite dimensional $\mathfrak{n}$-module, we will loosely write $L \in X$ if X contains the isomorphism class of L. Define an equivalence relation, $\sim$, on $\mathrm{Irr}_{\mathfrak{n}}^{f}$ as the smallest equivalence relation satisfying the following two conditions:

(I) For $L, S \in \mathrm{Irr}_{\mathfrak{n}}^{f}$ we have $L \sim S$ if there exists an indecomposable finite dimensional $\mathfrak{n}$-module M such that both $[M : L] \neq 0$ and $[M : S] \neq 0$.

(II) For $L, S \in \mathrm{Irr}_{\mathfrak{n}}^{f}$ we have $L \sim S$ if $[\mathfrak{g}/\mathfrak{n} \otimes L : S] \neq 0$.

Example 8. Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and $\mathfrak{n} = \mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Then $\mathfrak{h}$ is a commutative Lie algebra and hence $\mathrm{Irr}_{\mathfrak{n}}^{f}$ can be identified with the dual space $\mathfrak{h}^{*}$ in the natural way. Furthermore, if M is an indecomposable finite dimensional $\mathfrak{h}$-module, then, because of the commutativity of $\mathfrak{h}$, we have a decomposition

$$M = \oplus_{\lambda \in \mathfrak{h}^{*}} M_{\lambda}, \quad M_{\lambda} = \{v \in M : (h - \lambda(h))^{k} v = 0 \text{ for all } h \in \mathfrak{h}, k \gg 0\}$$

into a direct sum of $\mathfrak{h}$-modules. This means that the condition (I) only says $\lambda \sim \lambda$. The condition (II) gives $\lambda \sim \lambda + \alpha$ for any root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It follows that the equivalence class of $\mathfrak{h}^{*}$ with respect to $\sim$ has the form $\lambda + \mathbb{Z}\Delta$, where Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

For $I \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim$ and $M \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ denote by M_{I} the trace (i.e. the sum of all images) in M of all modules of the form $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}}N$, where N is a finite dimensional $\mathfrak{n}$-module such that for any simple finite dimensional $\mathfrak{n}$-module L we have that $[N : L] \neq 0$ implies $L \in I$.

Theorem 9. *Let* $M \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$.

(i) *For any $I \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim$ the vector space M_I is a $\mathfrak{g}$-submodule of M.*
   Moreover, for any simple finite dimensional $\mathfrak{n}$-module L we have
   $[M_I : L] \neq 0$ implies $L \in I$.

(ii) *We have $M = \oplus_{I \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim} M_I$.*

(iii) *If $I, J \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim$ and $I \neq J$ then $\mathrm{Hom}_{\mathfrak{g}}(M_I, M_J) = 0$.*

Proof. The vector space M_I is a $\mathfrak{g}$-submodule as the trace of some $\mathfrak{g}$-modules in any $\mathfrak{g}$-module is a $\mathfrak{g}$-submodule. Let N be a finite dimensional $\mathfrak{n}$-module such that for any simple finite dimensional $\mathfrak{n}$-module L we have that $[N : L] \neq 0$ implies $L \in I$. We claim that for any simple finite dimensional $\mathfrak{n}$-module L we even have that $[\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}} N : L] \neq 0$ implies $L \in I$.

First we recall that $\mathrm{Ind}_{\mathfrak{n}}^{\mathfrak{g}} N = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} N$, so we can use the PBW Theorem. Fix some basis in $\mathfrak{g}$ as described in the proof of Theorem 6 and for $n \in \mathbb{N}$ denote by $U(\mathfrak{g})_n$ the linear subspace of $U(\mathfrak{g})$, generated by all standard monomials of degree at most n. Then $U(\mathfrak{g})_n$ is an $\mathfrak{n}$-submodule of $U(\mathfrak{g})$ with respect to the adjoint action. Moreover, every finite dimensional $\mathfrak{n}$-submodule of the module $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} N$ is a submodule of the $\mathfrak{n}$-module $U(\mathfrak{g})_n \otimes_{U(\mathfrak{n})} N$ for some n. By [Di, 2.4.5] the $\mathfrak{g}$-module $U(\mathfrak{g})_n / U(\mathfrak{g})_{n-1}$ is isomorphic to the n-th symmetric power of $\mathfrak{g}$, which is a submodule of

$$\mathfrak{g}^{\otimes n} := \underbrace{\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n \text{ factors}}.$$

This implies that any simple subquotient of the $\mathfrak{n}$-module

$$U(\mathfrak{g})_n \otimes_{U(\mathfrak{n})} N / U(\mathfrak{g})_{n-1} \otimes_{U(\mathfrak{n})} N$$

is a subquotient of the module $(\mathfrak{g}/\mathfrak{n})^{\otimes n} \otimes N$. From the definition of the relation $\sim$ it follows by induction on n that any simple finite dimensional $\mathfrak{n}$-subquotient L of $(\mathfrak{g}/\mathfrak{n})^{\otimes n} \otimes N$ belongs to I. This proves the claim (i). The claim (iii) follows directly from the definitions and the claim (i).

To prove the claim (ii) let $X = \oplus_{I \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim} M_I$. Then X is a submodule of M by the claim (i). Consider a filtration of M of the form (1). As every M_i is finite dimensional, we obviously have $X \cap M_i = M_i$, which implies $X = M$. This completes the proof of the claim (ii) and of the whole theorem. $\qquad\square$

For $I \in \mathrm{Irr}_{\mathfrak{n}}^{f}/\sim$ denote by $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(I)$ the full subcategory of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$, which consists of modules M satisfying $M = M_I$. The categories $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(I)$ will be called *blocks* of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ and Theorem 9 gives a coproduct block decomposition of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$. From the category theoretical point of view it is enough to study each block $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(I)$ separately.

Example 10. Let $\mathfrak{g}$ and $\mathfrak{n}$ be as in Example 8. Then Theorem 9 gives the usual block decomposition for the category of weight modules with respect to supports of modules, indexed by cosets from $\mathfrak{h}^*/\mathbb{Z}\Delta$.

2.4. **$\mathfrak{n}$-socles of modules in $\mathfrak{W}_\mathfrak{n}^\mathfrak{g}$.** Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{g}$. Then every $M \in \mathfrak{W}_\mathfrak{n}^\mathfrak{g}$ has a well-defined $\mathfrak{n}$-socle $\mathrm{soc}_\mathfrak{n}(M)$. As usual we have the following standard result:

Proposition 11. *Let $M \in \mathfrak{W}_\mathfrak{n}^\mathfrak{g}$ and L be a simple finite dimensional $\mathfrak{n}$-module. Then*

$$[\mathrm{soc}_\mathfrak{n}(M) : L] = \dim \mathrm{Hom}_\mathfrak{n}(L, M) = \dim \mathrm{Hom}_\mathfrak{g}(\mathrm{Ind}_\mathfrak{n}^\mathfrak{g} L, M).$$

Proof. This follows from the usual adjunction between induction and restriction. □

Note that in the case $\mathfrak{g}/\mathfrak{n} \in \mathfrak{W}_\mathfrak{n}^\mathfrak{n}$, the rightmost homomorphism space in the formulation of Proposition 11 is inside the category $\mathfrak{W}_\mathfrak{n}^\mathfrak{g}$ by Theorem 6.

3. Whittaker modules revisited

3.1. **Finite dimensional representations of quasi-nilpotent Lie algebras.** For a Lie algebra $\mathfrak{n}$ define inductively ideals $\mathfrak{n}_0 := \mathfrak{n}$ and $\mathfrak{n}_i := [\mathfrak{n}_{i-1}, \mathfrak{n}]$, $i > 0$. Then we have a sequence of ideals

$$\mathfrak{n} = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \mathfrak{n}_2 \supset \dots.$$

We will say that $\mathfrak{n}$ is *quasi-nilpotent* provided that $\bigcap_{i=0}^{\infty} \mathfrak{n}_i = 0$. Obviously, any nilpotent Lie algebra is quasi-nilpotent. Until the end of this subsection we assume that $\mathfrak{n}$ is quasi-nilpotent. The next example of a quasi-nilpotent Lie algebra, which is no nilpotent, comes from the Virasoro algebra.

Example 12. Let $\mathfrak{n}$ have the basis $\{e_i : i = 1, 2, 3, \dots\}$ with the Lie bracket $[e_i, e_j] = (j - i)e_{i+j}$. Then it is easy to see that $\mathfrak{n}_k$, $k > 1$, is the subspace of $\mathfrak{n}$, spanned by $\{e_i : i = k + 2, k + 3, k + 4, \dots\}$ and hence $\mathfrak{n}$ is quasi-nilpotent.

Example 12 generalizes in a straightforward way to the positive part $\mathfrak{n}_+$ of any Lie algebra $\mathfrak{g}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of [MP].

The main for us property of quasi-nilpotent Lie algebras is the following easy fact:

Proposition 13. *Let $\mathfrak{n}$ be a quasi-nilpotent Lie algebra and M be a finite dimensional $\mathfrak{n}$-module. Then there is $i \in \mathbb{N}$ such that $\mathfrak{n}_i M = 0$.*

Proof. Let $\mathfrak{i}$ be the kernel of the Lie algebra homomorphism from $\mathfrak{n}$ to $\mathrm{End}_\mathbb{C}(M)$, defining the $\mathfrak{n}$-module structure on M. Then $\mathfrak{i}$ is an ideal of $\mathfrak{n}$ of finite codimension. Set $a_i = \dim(\mathfrak{n}_i + \mathfrak{i}/\mathfrak{i})$. As $\mathfrak{n}_{i+1} \subset \mathfrak{n}_i$, we have that the sequence $\mathbf{a} := \{a_i\}$ is weakly decreasing. As $\mathfrak{i}$ has finite codimension in $\mathfrak{n}$, all elements in $\mathbf{a}$ are finite positive integers. As

$\bigcap_{i=0}^{\infty} \mathfrak{n}_i = 0$, the sequence $\mathbf{a}$ converges to 1. This means that there exists $i \in \mathbb{N}$ such that $a_i = 1$, which means $\mathfrak{n}_i \subset \mathfrak{i}$ and thus $\mathfrak{n}_i M = 0$. This completes the proof. $\qquad\square$

Corollary 14. *Let $\mathfrak{n}$ be a quasi-nilpotent Lie algebra and L be a simple finite dimensional $\mathfrak{n}$-module. Then we have:*

   (i) $\dim L = 1$.
   (ii) $[\mathfrak{n}, \mathfrak{n}]L = 0$.

Proof. By Proposition 13, we have $\mathfrak{n}_i L = 0$ for some $i \in \mathbb{N}$. Hence L is a simple finite dimensional module over the nilpotent Lie algebra $\mathfrak{n}/\mathfrak{n}_i$. Now the claim (i) follows from the Lie Theorem ([Di, 1.3.13]). As $\dim L = 1$, we also have that the Lie algebra $\mathrm{End}_{\mathbb{C}}(L)$ is commutative, which implies the claim (ii). This completes the proof. $\qquad\square$

Set $\mathfrak{h} = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. Then $\mathfrak{h}$ is a commutative Lie algebra. From Corollary 14 it follows that $\mathrm{Irr}_{\mathfrak{n}}^f$ can be naturally identified with $\mathfrak{h}^*$ (compare with Example 8), which, in turn, can be identified with Lie algebra homomorphisms from $\mathfrak{n}$ to $\mathbb{C}$. In what follows we consider elements from $\mathfrak{h}^*$ as Lie algebra homomorphisms from $\mathfrak{n}$ to $\mathbb{C}$ under this identification. Now we would like to establish block decomposition for finite dimensional $\mathfrak{n}$-modules and show that the blocks are indexed by elements from $\mathfrak{h}^*$ in the natural way.

Let $\mathfrak{n}$ be a quasi-nilpotent Lie algebra and M be a finite dimensional $\mathfrak{n}$-module. For $\lambda \in \mathfrak{h}^*$ set

$$M_\lambda := \{v \in M : (x - \lambda(x))^k v = 0 \text{ for all } x \in \mathfrak{n} \text{ and } k \gg 0\}.$$

Proposition 15.   *(i) Each M_λ is a submodule of M.*
   (ii) $M = \oplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

Proof. Let $\mathfrak{m}$ denote the image of $\mathfrak{n}$ under the Lie algebra homomorphism defining the module structure of M. We can consider the algebra $\mathfrak{m}$ instead of the algebra $\mathfrak{n}$. Then, by Proposition 13, the algebra $\mathfrak{m}$ is a finite dimensional nilpotent Lie algebra and the claim follows from [Di, 1.3.19]. $\qquad\square$

As an immediate corollary we have the following:

Corollary 16. *Let $\mathfrak{n}$ be a quasi-nilpotent Lie algebra, M be an indecomposable finite dimensional $\mathfrak{n}$-module, and L be a simple submodule of M. Then $[M : L] = \dim M$. In particular, if S is a simple $\mathfrak{n}$-module such that $S \not\cong L$, then $[M : S] = 0$.*

Remark 17. Corollary 16 means that for quasi-nilpotent Lie algebras the condition (I) from the definition of the equivalence relation $\sim$ (see Subsection 2.3) trivializes (i.e. gives the equality relation).

3.2. **General Whittaker setup.** Now we are getting closer to the general situation in which one can consider Whittaker modules (see [Ko, OW, TZ, LW, Wa, LWZ, WZ]). We define the general Whittaker setup as follows: consider a Lie algebra $\mathfrak{g}$ and a quasi-nilpotent Lie subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ such that the action of $\mathfrak{n}$ on the adjoint $\mathfrak{n}$-module $\mathfrak{g}/\mathfrak{n}$ is locally nilpotent (in particular, $\mathfrak{g}/\mathfrak{n} \in \mathfrak{W}_\mathfrak{n}^\mathfrak{n}$). In this case we will say that $(\mathfrak{g}, \mathfrak{n})$ is a *Whittaker pair*. If $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair, then objects in $\mathfrak{W}_\mathfrak{n}^\mathfrak{g}$ will be called *Whittaker modules*.

Our terminology corresponds to the one used in [Ba, MS1, MS2]. In [Ko, OW] and many other papers a Whittaker module is additionally supposed to be generated by a Whittaker vector (see Section 4.1). The advantage of our definition is that the category of all Whittaker modules is abelian (see Subsection 2.1).

Example 18. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{z})$ is a Whittaker pair.

Example 19. Let $\mathfrak{n}$ be a quasi-nilpotent Lie algebra and $\mathfrak{a}$ be any Lie algebra. Then $(\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{n})$ is a Whittaker pair.

Example 20. Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathfrak{n}$ is nilpotent ([Di, 1.7.1]) and $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair.

Example 21. Let $\mathfrak{g}$ be a Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of [MP]. Let further $\mathfrak{n} = \mathfrak{n}_+$. Then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair. Note that in this case the algebra $\mathfrak{g}$ as well as the subalgebra $\mathfrak{n}$ may be infinite dimensional. This example contains, as special cases, situations studied in the articles [Ko, MD1, MD2, OW], where some simple Whittaker modules were described. It also includes many new examples, e.g. where $\mathfrak{g}$ is an affine Kac-Moody algebra.

Example 22. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{h}$ is commutative, in particular, it is quasi-nilpotent. However, $(\mathfrak{g}, \mathfrak{h})$ is *not* a Whittaker pair as the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ is not locally nilpotent.

Example 23. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. Let α be a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathfrak{n}$ be the corresponding root subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$. Then $\mathfrak{n}$ is an abelian subalgebra and $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair.

Example 24. Let $\mathfrak{g}$ be the Lie algebra with the basis

$$\{e_i : i \in \{\ldots, -2, -1, 0, 1\}\}$$

and the Lie bracket given by $[e_i, e_j] = (j - i)e_{i+j}$. Let $\mathfrak{n} = \langle e_1 \rangle$. Then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair. This example will be considered in more details in Section 5.

Example 25. Let $\mathfrak{g}$ be the Lie algebra with the basis

$$\{e_i : i \in \{-1, 0, 1, 2, \ldots\}\}$$

and the Lie bracket given by $[e_i, e_j] = (j - i)e_{i+j}$. Let $\mathfrak{n} = \langle e_1, e_2, \dots \rangle$ (see Example 12). Then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair. Simple Whittaker modules in this case were completely classified in [Ru].

Remark 26. It is easy to check that situations considered in the papers [TZ, LW, Wa, LWZ] also correspond to Whittaker pairs. Examples 24 and 25 generalize to any Witt algebra $\mathfrak{w}_n$, that is the Lie algebra of derivations of $\mathbb{C}[t_1, \dots, t_n]$ and more generally, to some other infinite dimensional Lie algebras of Cartan type (see also [Ru] for some partial results on Whittaker modules in these cases). The example of the algebra $\mathfrak{w}_n$ will be considered in more details in Section 5.

3.3. **Blocks for Whittaker pairs.** From now on, if not explicitly stated otherwise, we assume that $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair. Our first aim is to show that in this case the block decomposition of $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$, described in Subsection 2.3, trivializes in the following sense:

Theorem 27. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a Whittaker pair. Then the equivalence relation* $\sim$ *from Subsection 2.3 is the equality relation.*

Proof. From Remark 17 we already know that the condition (I) trivializes as $\mathfrak{n}$ is quasi-nilpotent. Hence we only have to check that the condition (II) trivializes as well.

Set $\mathfrak{h} = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ and identify $\mathfrak{h}^*$ with Lie algebra homomorphisms from $\mathfrak{n}$ to $\mathbb{C}$. For $\lambda \in \mathfrak{h}^*$ let L_λ denote the simple one-dimensional $\mathfrak{n}$-module, given by λ. Let v_λ be some fixed nonzero element in L_λ.

To prove the claim we have to show that for any $u \in \mathfrak{n}$ the element $u - \lambda(u)$ acts locally nilpotent on the $\mathfrak{n}$-module $\mathfrak{g}/\mathfrak{n} \otimes L_\lambda$. For any $w \in \mathfrak{g}/\mathfrak{n}$ we have

$$(u - \lambda(u))(w \otimes v_\lambda) = [u, w] \otimes v_\lambda + w \otimes u(v_\lambda) - \lambda(u)w \otimes v_\lambda = [u, w] \otimes v_\lambda,$$

as $(u - \lambda(u))v_\lambda = 0$. This implies, by induction, that

$$(2) \qquad\qquad (u - \lambda(u))^k(w \otimes v_\lambda) = \mathrm{ad}_u^k(w) \otimes v_\lambda.$$

As $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair, the adjoint action of any element from $\mathfrak{n}$ on the module $\mathfrak{g}/\mathfrak{n}$ is locally nilpotent. Hence $\mathrm{ad}_u^k(w) \otimes v_\lambda = 0$ for all $k \gg 0$, which implies $(u - \lambda(u))^k(w \otimes v_\lambda) = 0$ for all $k \gg 0$ by (2). The claim of the theorem follows. $\square$

Theorem 27 says that for any Whittaker pair $(\mathfrak{g}, \mathfrak{n})$ blocks of the category $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}$ of Whittaker modules, as defined in Subsection 2.3, are indexed by $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ in the natural way. We will denote these blocks by $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(\lambda)$. After Theorem 27 it is natural to say that the main problem in the theory of Whittaker modules is to describe the categories $\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(\lambda)$, $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$.

From Theorem 27 it follows that general Whittaker setup which leads to Whittaker modules is in some sense "opposite" to those pairs $(\mathfrak{g}, \mathfrak{n})$, for which one gets usual Harish-Chandra modules.

3.4. **Connection to Harish-Chandra subalgebras.** Recall (see [DFO]) that an associative unital algebra B is called *quasi-commutative* if $\mathrm{Ext}_B^1(L, S) = 0$ for any two nonisomorphic simple finite dimensional B-modules L and S. Let A be an algebra and B be a subalgebra of A. Following [DFO] we say that B is *quasi-central* provided that for any $a \in A$ the B-bimodule BaB is finitely generated both as a left and as a right B-module. A subalgebra B of A is called a *Harish-Chandra subalgebra* if it is both, quasi-commutative and quasi-central.

Theorem 28. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a Whittaker pair.*
  (i) *The algebra* $U(\mathfrak{n})$ *is quasi-commutative.*
  (ii) *If* $\dim \mathfrak{n} < \infty$, *then* $U(\mathfrak{n})$ *is a Harish-Chandra subalgebra of* $U(\mathfrak{g})$.

Proof. The claim (i) follows from Proposition 15. To prove (ii) we have only to show that $U(\mathfrak{n})$ is quasi-central. We would need the following variation of the PBW Theorem:

Lemma 29. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{n}$ *be a subalgebra of* $\mathfrak{g}$. *Let* $\{a_i : i \in I\}$ *(where* I *is well-ordered) be some basis in* $\mathfrak{n}$ *and* $\{b_j : j \in J\}$ *(where* J *is well-ordered) be a complement of* $\{a_i\}$ *to a basis of* $\mathfrak{g}$. *Then* $U(\mathfrak{g})$ *has a basis consisting of all elements of the form* $\mathbf{ba}$, *where* $\mathbf{b}$ *is a standard monomial in* $\{b_j\}$ *and* $\mathbf{a}$ *is a standard monomial in* $\{a_i\}$.

Proof. The proof is similar to the standard proof of the PBW Theorem and is left to the reader. $\square$

Lemma 30. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{n}$ *be a Lie subalgebra of* $\mathfrak{g}$ *such that the adjoint* $\mathfrak{n}$-*module* $\mathfrak{g}/\mathfrak{n}$ *belongs to* $\mathfrak{W}_{\mathfrak{n}}^{\mathrm{n}}$. *Then the adjoint* $\mathfrak{n}$-*module* $U(\mathfrak{g})/U(\mathfrak{n})$ *belongs to* $\mathfrak{W}_{\mathfrak{n}}^{\mathrm{n}}$ *as well.*

Proof. Choose some basis in $\mathfrak{g}$ as in Theorem 6 and the corresponding basis in $U(\mathfrak{g})$ as given by Lemma 29. Then the adjoint $\mathfrak{n}$-module $U(\mathfrak{g})/U(\mathfrak{n})$ can be identified with the linear span of standard monomials in $\{b_j\}$. Let $u \in U(\mathfrak{g})$. Then, writing u in our basis of $U(\mathfrak{g})$, we get a finite linear combination of standard monomials with some nonzero coefficients. In particular, only finitely many standard monomials in $\{b_j\}$ show up (as factors of summands in this linear combination). As $\mathfrak{g}/\mathfrak{n} \in \mathfrak{W}_{\mathfrak{n}}^{\mathrm{n}}$ by our assumption, applying the adjoint action of $\mathfrak{n}$ to all these standard monomials in $\{b_j\}$ we can produce, as summands, only finitely many new standard monomials in $\{b_j\}$. The claim follows. $\square$

For $u \in U(\mathfrak{n})$ consider the $U(\mathfrak{n})$-bimodule $X = U(\mathfrak{n})uU(\mathfrak{n})$, By Lemma 30, the image of X in $U(\mathfrak{g})/U(\mathfrak{n})$ is finite dimensional. At the same time $X \cap U(\mathfrak{n})$ is an ideal of $U(\mathfrak{n})$. As $\mathfrak{n}$ is finite dimensional, $U(\mathfrak{n})$ is noetherian ([Di, 2.3.8]). Hence $X \cap U(\mathfrak{n})$ is finitely generated as a left $U(\mathfrak{n})$-module. This implies that X is finitely generated as a left $U(\mathfrak{n})$-module. Applying the canonical antiinvolution on $\mathfrak{g}$ we obtain that X is finitely generated as a right $U(\mathfrak{n})$-module as well. Therefore $U(\mathfrak{n})$ is quasi-central and the claim (ii) of our theorem follows. $\square$

Remark 31. (a) There are natural examples of Whittaker pairs $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is finite dimensional while $\mathfrak{g}$ is infinite dimensional, see Example 24 and Remark 26.

(b) For finite dimensional $\mathfrak{n}$ to prove Lemma 30 one could alternatively argue using Propositions 4 and 1 and arguments similar to the ones used in the proof of Theorem 9.

(c) Theorem 6 follows from Lemma 30 and Propositions 4.

4. WHITTAKER VECTORS AND SIMPLE WHITTAKER MODULES

4.1. **Whittaker vectors, standard and simple Whittaker modules.** Let $(\mathfrak{g}, \mathfrak{n})$ be a Whittaker pair and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. As in the previous section we denote by L_λ the simple one-dimensional $\mathfrak{n}$-module given by λ. Let v_λ be some basis element of L_λ. Set $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} L_\lambda$ and call this module the *standard* Whittaker module.

Note that in [Ko, OW] and some other papers the module M_λ is called the universal Whittaker module, however the latter wording might be slightly misleading in our setup as not every Whittaker module is a quotient of M_λ (or direct sums of various M_λ's). As an example one could take $\mathfrak{g}$ to be a simple finite dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, $\mathfrak{n} = \mathfrak{n}_+$, $\lambda = 0$, and M any module in the BGG category $\mathcal{O}$, associated with this triangular decomposition, which is not generated by its highest weights (for example some projective module, which is not a Verma module).

Let $M \in \mathfrak{W}_\mathfrak{n}^\mathfrak{g}(\lambda)$. A vector $v \in M$ is called a *Whittaker vector* provided that $\langle v \rangle$ is an $\mathfrak{n}$-submodule of M (which is automatically isomorphic to L_λ by Theorem 27). Obviously, all Whittaker vectors form an $\mathfrak{n}$-submodule of M, which we will denote by $W_\lambda(M)$.

Proposition 32. *Let $M \in \mathfrak{W}_\mathfrak{n}^\mathfrak{g}(\lambda)$.*

(i) $\dim W_\lambda(M) = \dim \operatorname{Hom}_\mathfrak{g}(M_\lambda, M)$.

(ii) *If a Whittaker module M contains a unique (up to scalar) nonzero Whittaker vector, which, moreover, generates M, then M is a simple module.*

Proof. The claim (i) follows from Proposition 11. To prove the claim (ii) let $N \subset M$ be a nonzero submodule. Then N contains a nonzero Whittaker vector by Corollary 7. Since such vector in M is unique and generates M, we have $N = M$. This implies that M is simple and proves (ii). $\square$

Based on the examples from [Ko, OW, Ch, TZ] it looks reasonable to formulate the following conjectures:

Conjecture 33. Assume that $\mathfrak{g}$ is a Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of [MP]. Let L be a simple Whittaker module for the Whittaker pair $(\mathfrak{g}, \mathfrak{n}_+)$. Then $\operatorname{soc}_\mathfrak{n}(L)$ is a simple module.

Later on we will give some evidence for Conjecture 33 (in particular, in Subsection 5.4 we show that simple Whittaker modules with simple $\mathfrak{n}$-socle always exist). We note that Conjecture 33 does not extend to the general case, see example in Subsection 6.2.

Conjecture 34. Assume that $\mathfrak{g}$ is a Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of [MP], $\mathfrak{n} = \mathfrak{n}_+$ and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then for generic λ the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ surjects onto the set of Whittaker vectors of M_λ via $z \mapsto z \otimes v_\lambda$, $z \in Z(\mathfrak{g})$.

The Whittaker pair $(\mathfrak{g}, \mathfrak{n})$ associated with a fixed triangular decomposition of $\mathfrak{g}$ (see Example 21, Conjecture 34) seems to be the most reasonable situation to study Whittaker modules as it looks the most "balanced" one in the following sense: if the algebra $\mathfrak{n}$ is much "bigger" than $\mathfrak{g}/\mathfrak{n}$ then standard Whittaker modules should normally be simple (see for example some evidence for this in [Ru]); on the other hand if $\mathfrak{n}$ is much "smaller" than $\mathfrak{g}/\mathfrak{n}$ then standard Whittaker modules should normally have "too many" simple quotients with no chance of classifying them (take for example the situation described in Examples 19 and 23). The main advantage of special cases studied so far ([Ko, OW, Ch, TZ] and others) is that in those cases the considered situation was balanced enough to give a reasonable classification of generic simple Whittaker modules.

In the more general situation described in Section 2 already Conjecture 33 is not reasonable. In fact, as mentioned in Example 2, a special case of such situation is the study of weight modules over simple complex finite dimensional Lie algebras. At the same time, for such algebras there are many well-known examples of simple weight modules with infinitely many weights, such that all corresponding weight spaces are infinite dimensional (see for example [DFO]). In this case $\mathrm{soc}_\mathfrak{n}(L)$ has infinitely many nonisomorphic simple submodules, each occurring with infinite multiplicity.

Assume that $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair and that there exists a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$. Then from the PBW Theorem we have the decomposition $U(\mathfrak{g}) \cong U(\mathfrak{a}) \otimes U(\mathfrak{n})$ (as $U(\mathfrak{a})$-$U(\mathfrak{n})$-bimodules) and hence for any $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ the module M_λ can be identified with $U(\mathfrak{a})$ as a left $U(\mathfrak{a})$-module via the map

$$U(\mathfrak{a}) \xrightarrow{\varphi_\lambda} M_\lambda$$
$$u \mapsto u \otimes v_\lambda.$$

Proposition 35. *Assume that $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ for some subalgebra $\mathfrak{a}$. Let $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then $\varphi_\lambda^{-1}(W_\lambda(M_\lambda))$ is a subalgebra of $U(\mathfrak{a})$, isomorphic to $\mathrm{End}_\mathfrak{g}(M_\lambda)$.*

Proof. We have $\mathrm{End}_\mathfrak{g}(M_\lambda) = \mathrm{Hom}_\mathfrak{g}(M_\lambda, M_\lambda) \cong \mathrm{Hom}_\mathfrak{n}(L_\lambda, M_\lambda)$ by adjunction and $\mathrm{Hom}_\mathfrak{n}(L_\lambda, M_\lambda) \cong W_\lambda(M_\lambda)$ by the definition of $W_\lambda(M_\lambda)$. Now φ_λ^{-1} identifies elements of $W_\lambda(M_\lambda)$ with some elements from $U(\mathfrak{a})$.

As φ_λ is a homomorphism of $\mathfrak{a}$-modules, this identification is compatible with the product in $U(\mathfrak{a})$. The claim follows. $\square$

4.2. **Whittaker vectors in completions of highest and lowest weight modules.** In this subsection we assume that $\mathfrak{g}$ is a Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of [MP]. Set $\mathfrak{n} = \mathfrak{n}_+$. Then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair, see Example 21. For $\mu \in \mathfrak{h}^*$ consider the corresponding Verma module

$$M(\mu) := U(\mathfrak{g}) \bigotimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\mu,$$

where the $U(\mathfrak{h} \oplus \mathfrak{n}_+)$-module structure on $\mathbb{C}_\mu$ is defined as follows: $(h + n)(v) = \mu(h)v$, $h \in \mathfrak{h}$, $n \in \mathfrak{n}_+$, $v \in \mathbb{C}_\mu$, see [Di, Chapter 7], [MP, Section 2.3]. The module $M(\mu)$ is a highest weight module with highest weight μ and we have $M(\mu) = \oplus_{\nu \in \mathfrak{h}^*} M(\mu)_\nu$ (see notation of Example 8). Similarly we can define the corresponding lowest weight Verma module $N(\mu)$. From the definition we have that $N(\mu) \cong U(\mathfrak{n})$ as an $\mathfrak{n}$-module for any μ.

Consider also the completion

$$\overline{M(\mu)} := \prod_{\nu \in \mathfrak{h}^*} M(\mu)_\nu,$$

with the induced $\mathfrak{g}$-module structure. Our main result in this section is the following generalization of [Ko, Theorem 3.8]. The proof generalizes the original proof of [Ko, Theorem 3.8].

Theorem 36. *Assume that $\mu \in \mathfrak{h}^*$ is such that $M(\mu)$ is simple. Then for any $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ we have $\dim W_\lambda(\overline{M(\mu)}) = 1$, in particular, $W_\lambda(\overline{M(\mu)})$ is a simple $\mathfrak{n}$-module.*

Note that for Lie algebras with triangular decomposition the assumptions of Theorem 36 are generic (see [MP, Chapter 2]).

Proof. Using the canonical antiautomorphism $x \mapsto -x$ of $\mathfrak{g}$ we define on $N(-\mu)^*$ the structure of a $\mathfrak{g}$-module. As $N(-\mu)$ was a lowest weight module with lowest weight $-\mu$, the weight μ becomes a highest weight of $N(-\mu)^*$. In particular, from the universal property of Verma modules, we get a nonzero $\mathfrak{g}$-module homomorphism, say φ, from $M(\mu)$ to $N(-\mu)^*$. This homomorphism is, moreover, injective as the module $M(\mu)$ is simple by our assumptions.

Note that $\overline{M(\mu)} := \prod_{\nu \in \mathfrak{h}^*} M(\mu)_\nu$ and $N(-\mu)^* = \prod_{\nu \in \mathfrak{h}^*} N(-\mu)^*_{-\nu}$. Recall that $M(\mu) \cong U(\mathfrak{n}_-)$ as an $\mathfrak{n}_-$-module and $N(-\mu) \cong U(\mathfrak{n}_+)$ as an $\mathfrak{n}_+$-module. As $\mathfrak{g}$ has a triangular decomposition, the corresponding weight spaces in $U(\mathfrak{n}_+)$ and $U(\mathfrak{n}_-)$ have the same dimension. So the corresponding weight spaces in $M(\mu)$ and $N(-\mu)$ have the same dimension as well. From this it follows that φ extends in the obvious way to

an isomorphism $\overline{M(\mu)} \cong N(-\mu)^*$. The statement of the theorem now follows from the following lemma:

Lemma 37. *For any $\mu \in \mathfrak{h}^*$ we have $\dim W_\lambda(N(-\mu)^*) = 1$.*

Proof. Consider $N(-\mu)^*$ as a $U(\mathfrak{n})$-module, which we may identify with $U(\mathfrak{n})^*$. Denote by K_λ the kernel of the algebra homomorphism $U(\mathfrak{n}) \to \mathbb{C}$, induced by λ. For $f \in U(\mathfrak{n})^*$, $x \in \mathfrak{n}$ and $u \in U(\mathfrak{n})$ we have $((x - \lambda(x))f)(u) = f(-(x - \lambda(x))u)$. The latter is equal to zero for all $u \in U(\mathfrak{n})$ and $x \in \mathfrak{n}$ if and only if f annihilates K_λ. However, K_λ has, by definition, codimension one in $U(\mathfrak{n})$. It follows that $\dim W_\lambda(N(-\mu)^*) = 1$, which completes the proof. □

□

Let $\mu \in \mathfrak{h}^*$ and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. By Proposition 32(i) and Lemma 37, we have a unique homomorphism $\varphi : M_\lambda \to N(-\mu)^*$, which sends v_λ to a unique (up to scalar) nonzero vector in $W_\lambda(N(-\mu)^*)$. Let $L(\lambda, \mu)$ denote the image of this homomorphism. The following corollary from Theorem 36 and Lemma 37 gives some evidence for Conjecture 33:

Corollary 38. *Let $\mu \in \mathfrak{h}^*$ and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then $L(\lambda, \mu)$ is a simple $\mathfrak{g}$-module and satisfies $\dim W_\lambda(L(\lambda, \mu)) = 1$.*

Proof. The module $L(\lambda, \mu)$ is generated by v and satisfies $\dim W_\lambda(L(\lambda, \mu)) = 1$ by construction and Lemma 37. Simplicity of $L(\lambda, \mu)$ now follows from Proposition 32(ii). This completes the proof. □

In the case of the Virasoro algebra from the above results one obtains some new simple modules as we did not have any restriction on λ compared to [OW].

4.3. **Annihilators of $L(\lambda, \mu)$.** Here we continue to work in the setup of the previous subsection. Our main aim here is to prove the following theorem:

Theorem 39. *Assume that $\mu \in \mathfrak{h}^*$ is such that $M(\mu)$ is simple and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda, \mu) = \mathrm{Ann}_{U(\mathfrak{g})} M(\mu)$.*

Proof. Obviously, $\mathrm{Ann}_{U(\mathfrak{g})} M(\mu) \subset \mathrm{Ann}_{U(\mathfrak{g})} \overline{M(\mu)}$. As $L(\lambda, \mu) \subset \overline{M(\mu)}$ by construction, we thus have $\mathrm{Ann}_{U(\mathfrak{g})} M(\mu) \subset \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda, \mu)$.

To prove the reversed inclusion we use the arguments from the proof of [Ko, Theorem 3.9]. Let $X = \mathrm{Ann}_{U(\mathfrak{g})} M(\mu)$ and $Y = \mathrm{Ann}_{U(\mathfrak{g})} L(\lambda, \mu)$. Then both X and Y are stable with respect to the adjoint action of $\mathfrak{h}$. Assume $u \in Y$ and $\alpha \in \mathfrak{h}^*$ are such that $[h, u] = \alpha(h)u$ for all $h \in \mathfrak{h}$ and that $u \notin X$. Then there exists some $\xi \in \mathfrak{h}^*$ and $x \in M(\mu)_\xi$ such that $ux \neq 0$. Take any $y \in L(\lambda, \mu)$, $y \neq 0$, write it as an infinite sum $y = \sum_\nu y_\nu$ of weight vectors, and let y_ν be a nonzero summand. Then $y_\nu \in M(\mu)$ and hence, by the simplicity of $M(\mu)$, we have $x = ay_\nu$

for some weight element $a \in U(\mathfrak{g})$ (of weight $\xi - \nu$). We have $uay = \sum_\nu uay_\nu = 0$ as $ua \in Y$ and $y \in L(\lambda, \mu)$. On the other hand, the $\alpha + \xi$-component of uay is $ux \neq 0$. The obtained contradiction shows that $Y \subset X$, which completes the proof. $\square$

Conjecture 40. Let $\mu \in \mathfrak{h}^*$ be such that $M(\mu)$ is simple. Then for generic $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ the module $L(\lambda, \mu)$ is the unique (up to isomorphism) simple Whittaker module in $\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(\lambda)$, whose annihilator coincides with $\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda, \mu)$.

Note that, obviously, $L(\lambda, \mu) \cong L(\lambda', \mu')$ implies $\lambda = \lambda'$.

5. WHITTAKER MODULES OVER THE ALGEBRA OF DERIVATIONS OF $\mathbb{C}[x_1, x_2, \ldots, x_n]$

5.1. **The algebra $\mathfrak{w}_n$ and its decompositions.** Denote by $\mathfrak{w}_n$ the Lie algebra of all derivations of the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. For $i \in \{1, 2, \ldots, n\}$ and $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \{0, 1, 2, \ldots\}^n$ let

$$D_i(\mathbf{m}) := x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \frac{\partial}{\partial x_i} \in \mathfrak{w}_n.$$

Then $\mathbf{D} := \{D_i(\mathbf{m})\}$ is a natural basis of $\mathfrak{w}_n$. The algebra $\mathfrak{w}_n$ is a simple infinite dimensional Lie algebra of Cartan type with the Cartan subalgebra $\mathfrak{h}$ being the linear span of $x_i \frac{\partial}{\partial x_i}$, $i \in \{1, 2, \ldots, n\}$.

The linear span of the elements $x_i \frac{\partial}{\partial x_j}$, $i, j \in \{1, 2, \ldots, n\}$, is a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{w}_n$, isomorphic to $\mathfrak{gl}_n$. Note that $\mathfrak{h} \subset \mathfrak{a}$ is a Cartan subalgebra. For the rest we fix some triangular decomposition

$$\mathfrak{a} = \mathfrak{n}^{\mathfrak{a}}_- \oplus \mathfrak{h} \oplus \mathfrak{n}^{\mathfrak{a}}_+.$$

Denote by $\mathfrak{n}_+$ the linear span of $\mathfrak{n}^{\mathfrak{a}}_+$ and the elements $\frac{\partial}{\partial x_i}$, $i \in \{1, 2, \ldots, n\}$. Denote by $\mathfrak{n}_-$ the linear span of $\mathfrak{n}^{\mathfrak{a}}_-$ and all the elements $D_i(\mathbf{m})$, which are contained in neither $\mathfrak{a}$ nor $\mathfrak{n}_+$.

Proposition 41. *(i) We have the decomposition $\mathfrak{w}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ into a direct sum of Lie subalgebras.*
(ii) Both, $(\mathfrak{w}_n, \mathfrak{n}_-)$ and $(\mathfrak{w}_n, \mathfrak{n}_+)$, are Whittaker pairs.

Proof. The claim (i) is a straightforward calculation left to the reader. To prove the claim (ii) we have to verify that both $\mathfrak{n}_-$ and $\mathfrak{n}_+$ are quasi-nilpotent and that their actions on the corresponding adjoint modules $\mathfrak{w}_n/\mathfrak{n}_-$ and $\mathfrak{w}_n/\mathfrak{n}_+$ are locally nilpotent. This again can be done by a straightforward calculation. However, it is easier explained using the geometric picture of weights.

Let $\{e_i : i = 1, \ldots, n\}$ be the standard basis of $\mathbb{R}^n$. Then the weights of the adjoint action of $\mathfrak{h}$ on $\mathfrak{w}_n$ can be viewed as elements from $\mathbb{R}^n$ in the following way: the element $D_i(\mathbf{m})$ has weight $\mathbf{m} - e_i$. From our definition of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ it follows that there exists a hyperplane H in $\mathbb{R}^n$, containing 0 (the weight of $\mathfrak{h}$), such that the weights of $\mathfrak{n}_+$ and $\mathfrak{n}_-$

belong to different halfspaces with respect to H. Commutation of elements in $\mathfrak{w}_n$ corresponds, as usual, to the addition of respective weights (in fact, the claim (i) follows from this observation). As $\mathfrak{n}_+$ is finite dimensional, it follows that it must be nilpotent (as adding nonzero vectors we eventually would always leave the finite set of weights of $\mathfrak{n}_+$ in a finite number of steps). Similarly one shows that $\mathfrak{n}_-$ is quasi-nilpotent: commuting elements from $\mathfrak{n}_-$ we are moving the set of obtained weights further and further from the hyperplane H. In the limit, no weights will be left.

Now if we take some point in one of the halfspaces with respect to H and add to this point vectors from the other halfspace, representing weights of $\mathfrak{n}_+$ or $\mathfrak{n}_-$, respectively, we would always eventually obtain a point from the other halfspace. This shows that the action of $\mathfrak{n}_-$ and $\mathfrak{n}_+$ on the respective adjoint modules $\mathfrak{w}_n/\mathfrak{n}_-$ and $\mathfrak{w}_n/\mathfrak{n}_+$ is locally nilpotent. The claim of the proposition follows. $\square$

We would like to emphasize that the decomposition given by Proposition 41(i) is not a triangular decomposition in the sense of [MP] as the subalgebra $\mathfrak{n}_+$ is finite dimensional while the subalgebra $\mathfrak{n}_-$ is infinite dimensional.

The algebra $\mathfrak{w}_n$ has a subalgebra $\mathfrak{a}_1$, spanned by $\mathfrak{a}$, $\mathfrak{n}_+$, and the elements $x_j \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, $j \in \{1, 2, \ldots, n\}$. The algebra $\mathfrak{a}_1$ is isomorphic to $\mathfrak{sl}_{n+1}$ and $\mathfrak{h} \oplus \mathfrak{n}_+$ is a Borel subalgebra of $\mathfrak{a}_1$.

5.2. **Highest weight and lowest weight $\mathfrak{w}_n$-modules.** For $\mu \in \mathfrak{h}^*$ consider the simple $\mathfrak{h}$-module $\mathbb{C}_\mu = \mathbb{C}$ with the action given by $h(v) = \mu(h)v$, $h \in \mathfrak{h}$, $v \in \mathbb{C}_\mu$. Setting $\mathfrak{n}_+ \mathbb{C}_\mu = 0$ we extend $\mathbb{C}_\mu$ to a $\mathfrak{h} \oplus \mathfrak{n}_+$-module and can define the corresponding *highest weight Verma module*

$$M^+(\mu) := U(\mathfrak{w}_n) \bigotimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\mu.$$

As usual, the module $M^+(\mu)$ has a unique simple quotient, denoted by $L^+(\mu)$, which is a simple highest weight module of highest weight μ. As usual, simple highest weight modules are classified by their highest weights. Both $M^+(\mu)$ and $L^+(\mu)$ obviously have finite dimensional weight spaces.

Setting $\mathfrak{n}_- \mathbb{C}_\mu = 0$ we extend $\mathbb{C}_\mu$ to a $\mathfrak{h} \oplus \mathfrak{n}_-$-module and can define the corresponding *lowest weight Verma module*

$$M^-(\mu) := U(\mathfrak{w}_n) \bigotimes_{U(\mathfrak{h} \oplus \mathfrak{n}_-)} \mathbb{C}_\mu.$$

As usual, the module $M^-(\mu)$ has a unique simple quotient, denoted by $L^-(\mu)$, which is a simple lowest weight module of lowest weight μ. As usual, simple lowest weight modules are classified by their lowest

weights. Both $M^-(\mu)$ and $L^-(\mu)$ obviously have finite dimensional weight spaces.

Consider the full subcategory $\mathfrak{X}$ of the category of all $\mathfrak{w}_n$-modules, which consists of all weight (with respect to $\mathfrak{h}$) modules with finite dimensional weight spaces. For $M \in \mathfrak{X}$ we can write $M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu$, where each M_μ is finite dimensional. Then $M^\star := \oplus_{\mu \in \mathfrak{h}^*} M_\mu^*$, which is a subspace of the full dual space M^*, becomes a $\mathfrak{w}_n$-module via $(xf)(v) := f(-x(v))$ for $x \in \mathfrak{w}_n$, $f \in M^\star$ and $v \in M$. As usual, this extends to an exact contravariant and involutive self-equivalence $\star$ on the category $\mathfrak{X}$.

Proposition 42. *For any $\mu \in \mathfrak{h}^*$ we have*

$$L^+(\mu)^\star \cong L^-(-\mu) \ and \ L^-(-\mu)^\star \cong L^+(\mu).$$

Proof. As $\star$ is defined using the canonical involution $x \mapsto -x$ on $\mathfrak{w}_n$, it sends a module with highest weight μ to a module with lowest weight $-\mu$ and vice versa. As $\star$ is a self-equivalence, it sends simple modules to simple modules. The claim follows. $\square$

We emphasize the following immediate corollary, which is interesting as the analogous equality is certainly wrong for the corresponding Verma modules (since the algebra $\mathfrak{n}_-$ is much "bigger" than the algebra $\mathfrak{n}_+$):

Corollary 43. *For all $\mu, \nu \in \mathfrak{h}^*$ we have $\dim L^+(\mu)_\nu = \dim L^-(-\mu)_{-\nu}$.*

Proposition 44. *The modules $M^-(\mu)$ are generically irreducible.*

Proof. The restriction of $M^-(\mu)$ to $\mathfrak{a}_1$ is a Verma module, which is generically irreducible. $\square$

Note that the $\mathfrak{w}_n$-module $M^-(\mu)$ may be irreducible even if its restriction to $\mathfrak{a}_1$ is reducible. We refer the reader to [Ru] for more details.

5.3. **Whittaker modules for $\mathfrak{w}_n$.** For the Whittaker pair $(\mathfrak{w}_n, \mathfrak{n}_-)$ simple Whittaker modules form a subclass of modules, considered in [Ru]. In this subsection we generalize the construction of Whittaker modules from Subsection 4.2 to obtain simple Whittaker modules for the Whittaker pair $(\mathfrak{w}_n, \mathfrak{n}_+)$. We set $\mathfrak{n} := \mathfrak{n}_+$. Note one big difference with the setup of Subsection 4.2: the decomposition of the algebra $\mathfrak{w}_n$ we work with (Subsection 5.1) is not a triangular decomposition in the sense of [MP].

For $\mu \in \mathfrak{h}^*$ we have a decomposition $L^+(\mu) = \oplus_{\nu \in \mathfrak{h}^*} L^+(\mu)_\nu$. Consider the corresponding completion

$$\overline{L^+(\mu)} = \prod_{\nu \in \mathfrak{h}^*} L^+(\mu)_\nu$$

of $L^+(\mu)$, which becomes a $\mathfrak{w}_n$-module in the natural way.

Theorem 45. *Assume that $\mu \in \mathfrak{h}^*$ is such that $M^-(-\mu)$ is simple. Then for any $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ we have $\dim W_\lambda(\overline{L^+(\mu)}) = 1$, in particular, $W_\lambda(\overline{L^+(\mu)})$ is a simple $\mathfrak{n}$-module.*

Proof. Mutatis mutandis the proof of Theorem 36. Note that we have $M^-(-\mu) = L^-(-\mu)$ under our assumptions, however $M^+(\mu) \not\cong L^+(\mu)$. Thus in all arguments from the proof of Theorem 36 one should consider $L^+(\mu)$ instead of the corresponding Verma module. $\square$

For μ and λ as in Theorem 45 let $L(\lambda, \mu)$ denote the image of the unique (up to scalar) nonzero homomorphism from M_λ to $\overline{L^+(\mu)}$. Similarly to Corollary 38 we have:

Corollary 46. *Assume that $\mu \in \mathfrak{h}^*$ is such that $M^-(-\mu)$ is simple and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then $L(\lambda, \mu)$ is a simple $\mathfrak{w}_n$-module and satisfies $\dim W_\lambda(L(\lambda, \mu)) = 1$.*

Proof. Mutatis mutandis the proof of Corollary 38. $\square$

Corollary 47. *Assume that $\mu \in \mathfrak{h}^*$ is such that $M^-(-\mu)$ is simple and $\lambda \in (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Then $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda, \mu) = \operatorname{Ann}_{U(\mathfrak{g})} L^+(\mu)$.*

Proof. Mutatis mutandis the proof of Theorem 39. $\square$

It is worth to mention that annihilators of simple highest weight $\mathfrak{w}_1$-modules are described in [CM]. Note that the restriction of the module $L(\lambda, \mu)$, constructed in Theorem 45, to the algebra $\mathfrak{a}_1$ obviously has a simple socle (as $\dim W_\lambda(L(\lambda, \mu)) = 1$), which is a simple Whittaker module for the Whittaker pair $(\mathfrak{a}_1, \mathfrak{n}_+)$. We finish with the following conjecture:

Conjecture 48. The modules $L(\lambda, \mu)$, $\lambda \neq 0$, constructed in Theorem 45, constitute an exhaustive list of simple Whittaker modules with $\lambda \neq 0$ for the Whittaker pair $(\mathfrak{w}_n, \mathfrak{n}_+)$. Moreover, $L(\lambda, \mu) \cong L(\lambda, \mu')$ if and only if $\operatorname{Ann}_{U(\mathfrak{g})} M^-(-\mu) = \operatorname{Ann}_{U(\mathfrak{g})} M^-(-\mu')$.

5.4. **A general existence theorem.** The arguments used in the proof of Theorems 36 and 45 can be easily generalized to prove the following general existence theorem:

Theorem 49. *Let $(\mathfrak{g}, \mathfrak{n})$ be a Whittaker pair and assume that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$ for some Lie subalgebra $\mathfrak{a}$. Then for every $\lambda : \mathfrak{n} \to \mathbb{C}$ there exists a (simple) module $M \in \mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(\lambda)$ such that $W_\lambda(M)$ is one-dimensional.*

Proof. Let L be any one-dimensional $\mathfrak{a}$-module. Then the $\mathfrak{a}$-module $\operatorname{Ind}_{\mathfrak{a}}^{\mathfrak{g}}(L)$ is isomorphic to $\mathfrak{n}$ as $\mathfrak{n}$-module. Analogously to the proofs of Theorems 36 and 45 we get that $W_\lambda(\operatorname{Ind}_{\mathfrak{a}}^{\mathfrak{g}}(L)^*)$ is one-dimensional. The claim follows. $\square$

6. Further examples: Whittaker modules over solvable finite dimensional Lie algebras

6.1. **The two-dimensional solvable algebra.** Consider the two-dimensional Lie algebra $\mathfrak{g}$ with the basis $\{a, b\}$ and the Lie bracket given by $[a, b] = b$. Let $\mathfrak{n} = \langle b \rangle$, then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair (Example 20). All simple modules over the algebra $\mathfrak{g}$ are described in [Bl], however, we are also interested in the description of the category of Whittaker modules (at least those of finite length). For $\lambda \in \mathbb{C}$ we consider the standard Whittaker module $M_\lambda = U(\mathfrak{g})/(b - \lambda)$.

Proposition 50. *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$.*

(i) *The module M_λ is irreducible and hence is a unique (up to isomorphism) simple object in $\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(\lambda)$.*

(ii) *We have*

$$\mathrm{Ext}_\mathfrak{g}^i(M_\lambda, M_\lambda) \cong \begin{cases} \mathbb{C}, & i = 0; \\ 0, & i > 0, \end{cases}$$

in particular, the category $\overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(\lambda)}$ is equivalent to $\mathbb{C}$-mod.

Proof. The claim (i) follows from [Bl]. Here is a short argument: By the PBW Theorem we have $M_\lambda \cong \mathbb{C}[a]$ as a $\mathbb{C}[a]$-module. In particular, M_λ is generated by 1 as a $\mathbb{C}[a]$-module. The action of b on $\mathbb{C}[a]$ is given by $b \cdot f(a) = \lambda f(a-1)$, $f(a) \in \mathbb{C}[a]$. Assume that M_λ is not simple and let N be a proper submodule of M_λ. Let $f(a) \in N$ be a nonzero element of minimal degree. Then $\deg f(a) = k > 0$ as M_λ is generated by scalars already as a $\mathbb{C}[a]$-module. For the element $f(a) - \frac{1}{\lambda} b \cdot f(a) \in N$ we have

$$\deg(f(a) - \frac{1}{\lambda} b \cdot f(a)) = \deg(f(a) - f(a-1)) = k - 1,$$

a contradiction. The claim (i) follows.

To prove the claim (ii) we consider the following free resolution of M_λ:

$$0 \to U(\mathfrak{g}) \xrightarrow{\cdot (b-\lambda)} U(\mathfrak{g}) \to U(\mathfrak{g})/(b - \lambda) \cong M_\lambda \to 0.$$

Applying $\mathrm{Hom}_{U(\mathfrak{g})}(\_, M_\lambda)$ we get

$$(3) \qquad\qquad 0 \to M_\lambda \xrightarrow{(b-\lambda)\cdot} M_\lambda \to 0.$$

This implies that $\mathrm{Ext}_\mathfrak{g}^i(M_\lambda, M_\lambda) = 0$ for all $i > 1$. Further, we have that $\mathrm{Ext}_\mathfrak{g}^1(M_\lambda, M_\lambda)$ is isomorphic to the cokernel of the linear operator $(b - \lambda)\cdot$ on M_λ. We claim that this cokernel is zero. Indeed, for $f(a) \in \mathbb{C}[a]$ we have

$$(b - \lambda) \cdot f(a) = \lambda(f(a-1) - f(a)).$$

As $\lambda \neq 0$, the cokernel of $(b-\lambda)\cdot$ equals the cokernel of the linear operator $f(a) \mapsto f(a-1) - f(a)$. This cokernel is obviously zero. Therefore

$\mathrm{Ext}^1_{\mathfrak{g}}(M_\lambda, M_\lambda) = 0$ and hence the category $\overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(\lambda)}$ is semisimple. This proves the claim (ii) and completes the proof of the proposition. $\square$

For $\mu \in \mathbb{C}$ denote by $L(\mu)$ the one-dimensional $\mathfrak{g}$-module given by $b \cdot L(\mu) = 0$ and $a \cdot v = \mu v$, $v \in L(\mu)$.

Proposition 51. *(i) The modules $\{L(\mu) : \mu \in \mathbb{C}\}$ constitute an exhaustive and irredundant list of pairwise nonisomorphic simple objects in $\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)$.*

(ii) For $\mu, \nu \in \mathbb{C}$ we have

(4)
$$\mathrm{Ext}^1_{\mathfrak{g}}(L(\mu), L(\nu)) \cong \begin{cases} \mathbb{C}, & \nu \in \{\mu, \mu+1\}; \\ 0, & otherwise; \end{cases}$$

(5)
$$\mathrm{Ext}^2_{\mathfrak{g}}(L(\mu), L(\nu)) \cong \begin{cases} \mathbb{C}, & \nu = \mu+1; \\ 0, & otherwise; \end{cases}$$

(6)
$$\mathrm{Ext}^i_{\mathfrak{g}}(L(\mu), L(\nu)) \cong 0, \quad i > 2.$$

Proof. We have $b \cdot M_0 = 0$ and hence from Proposition 32(i) it follows that $b \cdot L = 0$ for any simple $L \in \mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)$. Therefore any simple object in $\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)$ is, in fact, a simple module over the polynomial algebra $\mathbb{C}[a]$. The claim (i) follows.

For $\mu \in \mathbb{C}$ it is easy to see that the following is a free resolution of the module $L(\mu)$:

$$0 \to U(\mathfrak{g}) \xrightarrow{\left(\begin{array}{c} \cdot(a-\mu-1) \\ \cdot b \end{array} \right)} U(\mathfrak{g}) \oplus U(\mathfrak{g}) \xrightarrow{(-\cdot b, \cdot(a-\mu))} U(\mathfrak{g}) \to$$
$$\to U(\mathfrak{g})/(a - \mu, b) \cong L(\mu) \to 0.$$

In particular, all extensions of degree three and higher vanish and the formula (6) follows. Applying $\mathrm{Hom}_{U(\mathfrak{g})}(\_, L(\nu))$, $\nu \in \mathbb{C}$, we get

(7)
$$0 \to L(\nu) \xrightarrow{\left(\begin{array}{c} -b\cdot \\ (a-\mu)\cdot \end{array} \right)} L(\nu) \oplus L(\nu) \xrightarrow{((a-\mu-1)\cdot, b\cdot)} L(\nu) \to 0.$$

As $L(\nu) \cong \mathbb{C}$ and $b \cdot L(\nu) = 0$, in the case $\nu \notin \{\mu, \mu+1\}$ we immediately obtain that (7) is exact. If $\nu = \mu$, we have one dimensional homologies in degrees zero and one. If $\nu = \mu + 1$, we have one dimensional homologies in degrees one and two. This gives the formulae (4) and (5). The claim (ii) follows and the proof is complete. $\square$

Now we would like to determine a decomposition of the category $\overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)}$ into indecomposable subcategories. For $\xi \in \mathbb{C}/\mathbb{Z}$ denote by $\overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)}_\xi$ the full subcategory of $\overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)}$, which consists of all modules, whose simple subquotients have the form $L(\mu)$, $\mu \in \xi$.

Theorem 52. *(i)* $\overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)} \cong \bigoplus_{\xi \in \mathbb{C}/\mathbb{Z}} \overline{\mathfrak{W}^{\mathfrak{g}}_{\mathfrak{n}}(0)}_\xi.$

(ii) Each $\overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(0)}_\xi$ is equivalent to the category of finite-dimensional modules over the following quiver:

(8)

with relations $\mathsf{b}_i\mathsf{a}_i = \mathsf{a}_{i+1}\mathsf{b}_i$ for all i, where every a_i acts locally nilpotent.

Proof. If $\xi \in \mathbb{C}/\mathbb{Z}$ and $\mu \notin \xi$, then from Proposition 51(ii) we have that the first extension from $L(\mu)$ to any $L(\nu)$, $\nu \in \xi$, an vice versa, vanishes. The claim (i) follows.

Fix now $\xi \in \mathbb{C}/\mathbb{Z}$. As $\mathfrak{n}$ is finite dimensional, the category $\mathfrak{W}_\mathfrak{n}^\mathfrak{g}$ is extension closed in $\mathfrak{g}$-Mod by Proposition 1. Hence, by Proposition 51(ii), the quiver given in (ii) is the ext-quiver of $\overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(0)}_\xi$. The relation

(9) $$\mathsf{b}_i\mathsf{a}_i = \mathsf{a}_{i+1}\mathsf{b}_i$$

follows immediately from the relation $ab = b(a+1)$ in $U(\mathfrak{g})$.

Assume that $w = 0$ is a new relation for $\overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(0)}_\xi$, which does not follow from the relations (9), and that the maximal degree of a monomial in w is $k-1$ for some $k \in \mathbb{N}$.

Consider the full subcategory $\mathfrak{X}$ of $\overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(0)}_\xi$, consisting of all M such that $b^k \cdot M = 0$ and such that the minimal polynomial of the action of a on M has roots of multiplicities at most k.

For $k \in \mathbb{Z}$ and $\mu \in \xi$ consider the $\mathfrak{g}$-module

$$V_k(\mu) = U(\mathfrak{g})/((a-\mu)^k, b^k) \in \overline{\mathfrak{W}_\mathfrak{n}^\mathfrak{g}(0)}_\xi.$$

It is easy to see that the endomorphism algebra of this module is isomorphic to $\mathbb{C}[x]/(x^k)$ (acting via multiplication with $a - \mu$ from the right). In particular, the module $V_k(\mu)$ is indecomposable for all k and is generated by 1. It is easy to see that $V_k(\mu)$ is, in fact, an indecomposable projective in $\mathfrak{X}$. From the PBW Theorem we have that $\dim V_k(\mu) = k^2$.

Consider the quotient A of our quiver algebra (8), given by (9) and additional relations (for all i)

(10) $$\mathsf{a}_i^k = 0, \quad \mathsf{b}_{i+k-1}\cdots\mathsf{b}_{i+1}\mathsf{b}_i = 0.$$

Then all indecomposable projective A-modules have dimension k^2. If we would add the additional relation $w = 0$ (which is not a consequence of (9) and (10) by our choice of k), the dimension of indecomposable projectives A-modules would decrease. This, however, contradicts the previous paragraph. The claim of the theorem follows. $\square$

The quiver algebra, described in Theorem 52(ii), is Koszul (we refer the reader to [MOS] for Koszul theory for algebras with infinitely many simple modules). In particular, there is a graded version of the

category $\overline{\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(0)}_{\xi}$, which is equivalent to the category of finite dimensional modules over the following quiver subject to the relations that all squares commutes:
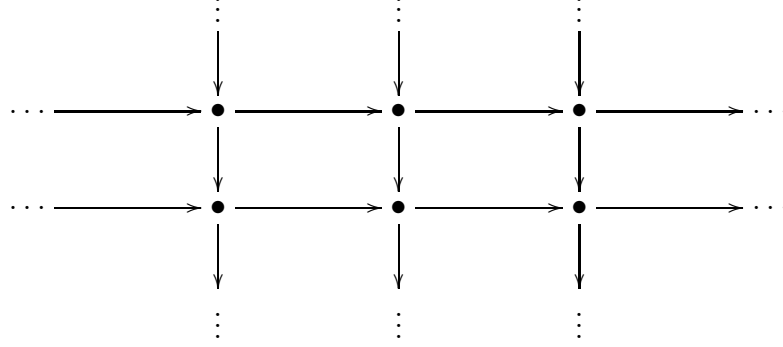
This is interesting as a priori we do not have any grading on $\overline{\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{g}}(0)}_{\xi}$.

6.2. **A three-dimensional nilpotent algebra.** Let $\mathfrak{g}$ denote the three-dimensional Lie algebra with the basis $\{a, b, c\}$, where the Lie bracket is given by

$$[a, b] = c, \quad [a, c] = 0, \quad [b, c] = 0.$$

Let $\mathfrak{n} = \langle c \rangle$, then $(\mathfrak{g}, \mathfrak{n})$ is a Whittaker pair (Example 20). As c is central it acts as a scalar on any simple $\mathfrak{g}$-module ([Di, 2.6.8]). Hence any simple $\mathfrak{g}$-module is a Whittaker module for the Whittaker pair $(\mathfrak{g}, \mathfrak{n})$ (in particular, it might have many Whittaker vectors, compare with Conjecture 33). For every $\theta \in \mathbb{C}$ the quotient $U(\mathfrak{g})/(c - \theta)$ is isomorphic to the first Weyl algebra A_1. Simple modules over this algebra are described in [Bl] and the blocks for this algebra seem to be very complicated, see for example [Bav].

6.3. **Borel subalgebras.** Let $\mathfrak{g}$ be a semi-simple finite dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is a Borel subalgebra of $\mathfrak{g}$ and $(\mathfrak{b}, \mathfrak{n})$, where $\mathfrak{n} = \mathfrak{n}_+$, is a Whittaker pair (Example 20). Note that the Whittaker pair considered in Subsection 6.1 was a special case of the present situation ($\mathfrak{g} = \mathfrak{sl}_2$). Let Δ_+ be the set of positive roots for $\mathfrak{g}$ (that is roots for $\mathfrak{n}$) and π be the set of simple roots. For $\alpha \in \Delta_+$ we fix some nonzero root vector $e_\alpha \in \mathfrak{g}_\alpha \subset \mathfrak{n}$. Let h_α, $\alpha \in \pi$, be the basis of $\mathfrak{h}$, dual to π. We have $[\mathfrak{n}, \mathfrak{n}] = \langle e_\alpha : \alpha \in \Delta_+ \setminus \pi \rangle$. In particular, every Lie algebra homomorphism $\lambda : \mathfrak{n} \to \mathbb{C}$ satisfies $\lambda(e_\alpha) = 0$, $\alpha \in \Delta_+ \setminus \pi$, and is uniquely defined by $\lambda_\alpha = \lambda(e_\alpha)$, $\alpha \in \pi$.

For $\lambda : \mathfrak{n} \to \mathbb{C}$ set $\pi_\lambda = \{\alpha \in \pi : \lambda_\alpha \neq 0\}$. Define

$$\mathfrak{h}^\lambda = \langle h_\alpha : \alpha \in \pi_\lambda \rangle, \qquad \mathfrak{h}_\lambda = \langle h_\alpha : \alpha \in \pi \setminus \pi_\lambda \rangle.$$

For $\lambda : \mathfrak{n} \to \mathbb{C}$ and $\mu \in \mathfrak{h}_\lambda^*$ let $I_{\lambda,\mu}$ denote the left ideal of $U(\mathfrak{b})$ generated by the elements $e_\alpha - \lambda_\alpha$, $\alpha \in \pi$, and $h - \mu(h)$, $h \in \mathfrak{h}_\lambda$. Set

$L_{\lambda,\mu} = U(\mathfrak{b})/I_{\lambda,\mu}$. From the PBW Theorem we have that $U(\mathfrak{b})/I_{\lambda,\mu} \cong \mathbb{C}[\mathfrak{h}^\lambda]$ as a $\mathbb{C}[\mathfrak{h}^\lambda]$-module.

Proposition 53. *The modules $\{L_{\lambda,\mu} : \lambda : \mathfrak{n} \to \mathbb{C}, \mu \in \mathfrak{h}_\lambda^*\}$ constitute an exhaustive and irredundant list of simple modules in $\mathfrak{W}_\mathfrak{n}^\mathfrak{b}$.*

Proof. From the definition it follows easily that the canonical generator of $L_{\lambda,\mu}$ is a unique (up to scalar) Whittaker element of $L_{\lambda,\mu}$. Hence simplicity of $L_{\lambda,\mu}$ follows from Proposition 32(ii). That these modules are pairwise non-isomorphic follows directly from the definition.

For a fixed $\lambda : \mathfrak{n} \to \mathbb{C}$ let $L \in \mathfrak{W}_\mathfrak{n}^\mathfrak{b}(\lambda)$ be simple. Then L is a quotient of M_λ by Proposition 32(i), in particular, $e_\alpha L = 0$ for any $\alpha \in \Delta_+ \setminus \pi_\lambda$. Hence that action of $U(\mathfrak{h}_\lambda)$ on L commutes with the action of the whole $U(\mathfrak{b})$ and thus gives endomorphisms of L. However, every endomorphism of a simple module reduces to scalars by [Di, 2.6.4], so every element of $U(\mathfrak{h}_\lambda)$ acts on L as a scalar, say the one, given by $\mu \in \mathfrak{h}_\lambda^*$. This implies that $I_{\lambda,\mu}$ annihilates the Whittaker vector of L. The latter means that L is a nonzero quotient of $L_{\lambda,\mu}$ and hence $L \cong L_{\lambda,\mu}$ as $L_{\lambda,\mu}$ is simple. This completes the proof. $\square$

For every $\alpha \in \pi$ the space $\mathfrak{a}_\alpha = \langle h_\alpha, e_\alpha \rangle$ is a Lie subalgebra of $\mathfrak{b}$, isomorphic to the Lie algebra $\mathfrak{g}$ from Subsection 6.1 and we have

$$\bigoplus_{\alpha \in \pi} \mathfrak{a}_\alpha \cong \mathfrak{a} := \mathfrak{b}/[\mathfrak{n}, \mathfrak{n}].$$

For $\lambda : \mathfrak{n} \to \mathbb{C}$ and $\mu \in \mathfrak{h}_\lambda^*$ we have $[\mathfrak{n}, \mathfrak{n}]M_\lambda = 0$ from the definition and hence $[\mathfrak{n}, \mathfrak{n}]L_{\lambda,\mu} = 0$ as well. This makes $L_{\lambda,\mu}$ an $\mathfrak{a}$-module and we have an obvious isomorphism of $\mathfrak{a}$-modules:

$$(11) \qquad L_{\lambda,\mu} \cong \bigotimes_{\alpha \in \pi_\lambda} L_{\lambda_\alpha}^{\mathfrak{a}_\alpha} \otimes \bigotimes_{\alpha \in \pi \setminus \pi_\lambda} L^{\mathfrak{a}_\alpha}(\mu(h_\alpha))$$

(here the superscript $\mathfrak{a}_\alpha$ means that the module in question is an $\mathfrak{a}_\alpha$-module and the notation is as in Subsection 6.1). This isomorphism extends to $\mathfrak{b}$-modules using the trivial action of $[\mathfrak{n}, \mathfrak{n}]$. To simplify notation we set

$$L_{\lambda,\mu}^{\mathfrak{a}_\alpha} := \begin{cases} L_{\lambda_\alpha}^{\mathfrak{a}_\alpha}, & \alpha \in \pi_\lambda; \\ L^{\mathfrak{a}_\alpha}(\mu(h_\alpha)), & \alpha \in \pi \setminus \pi_\lambda. \end{cases}$$

Proposition 54. *Let $\lambda : \mathfrak{n} \to \mathbb{C}$, $\mu \in \mathfrak{h}_\lambda^*$, $\lambda' : \mathfrak{n} \to \mathbb{C}$, $\mu' \in \mathfrak{h}_\lambda^*$. Then*

$$\mathrm{Ext}_\mathfrak{a}^k(L_{\lambda,\mu}, L_{\lambda',\mu'}) \cong \sum_{\substack{(i_\alpha) \in \{0,1,2,\dots\}^\pi \\ \sum_\alpha i_\alpha = k}} \bigotimes_{\alpha \in \pi} \mathrm{Ext}_{\mathfrak{a}_\alpha}^{i_\alpha}(L_{\lambda,\mu}^{\mathfrak{a}_\alpha}, L_{\lambda',\mu'}^{\mathfrak{a}_\alpha}),$$

in particular, all these extension spaces are finite dimensional.

We note that all extension spaces on the right hand side of the above formula are explicitly described in Subsection 6.1.

Proof. As $\mathfrak{a} \cong \oplus_{\alpha \in \pi} \mathfrak{a}_\alpha$ and $U(\mathfrak{a}) \cong \otimes_{\alpha \in \pi} U(\mathfrak{a}_\alpha)$, the claim follows from (11) and the Künneth formula. $\square$

Corollary 55. *All vector spaces* $\mathrm{Ext}^1_{\mathfrak{b}}(L_{\lambda,\mu}, L_{\lambda',\mu'})$ *are finite dimensional.*

Proof. To compute the extension spaces in question we use the classical cohomology of Lie algebras ([CE, Chapter XIII]). Consider the Lie algebra $\mathfrak{c} = \mathfrak{n} \oplus \mathfrak{h}_\lambda$ and observe that $L_{\lambda,\mu}$ is induced from the one-dimensional $\mathfrak{c}$-module $\mathsf{L}_{\lambda,\mu}$, given by λ and μ, by construction. We denote by $\nu : \mathfrak{c} \to \mathbb{C}$ the Lie algebra homomorphism, defining $\mathsf{L}_{\lambda,\mu}$. Consider the first three steps of the free resolution of the trivial $\mathfrak{c}$-module $\mathbb{C}$:

$$(12) \qquad U(\mathfrak{c}) \otimes \mathfrak{c} \wedge \mathfrak{c} \xrightarrow{\gamma_1} U(\mathfrak{c}) \otimes \mathfrak{c} \xrightarrow{\beta_1} U(\mathfrak{c}) \otimes \mathbb{C} \xrightarrow{\alpha_1} \mathbb{C} \to 0,$$

where the map α is given by $1 \otimes 1 \mapsto 1$; the map β is given by $1 \otimes x \mapsto x$, $x \in \mathfrak{c}$; and the map γ is given by

$$1 \otimes x \wedge y \mapsto x \otimes y - y \otimes x - 1 \otimes [x,y], \qquad x, y \in \mathfrak{c}.$$

Tensoring (12) with the $\mathfrak{c}$-module $\mathsf{L}_{\lambda,\mu}$ (over $\mathbb{C}$) and further with $U(\mathfrak{b})$ over $U(\mathfrak{c})$ (which is exact by the PBW Theorem), we obtain a free $U(\mathfrak{b})$-resolution of $L_{\lambda,\mu} \cong U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (\mathbb{C} \otimes \mathsf{L}_{\lambda,\mu})$ as follows:

(13)

$$U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathfrak{c} \wedge \mathfrak{c} \otimes \mathsf{L}_{\lambda,\mu}) \xrightarrow{\gamma_2} U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathfrak{c} \otimes \mathsf{L}_{\lambda,\mu}) \xrightarrow{\beta_2}$$

$$\xrightarrow{\beta_2} U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathbb{C} \otimes \mathsf{L}_{\lambda,\mu}) \xrightarrow{\alpha_2} U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (\mathbb{C} \otimes \mathsf{L}_{\lambda,\mu}) \to 0,$$

where $x_2 = \mathrm{id} \otimes (x_1 \otimes \mathrm{id})$, $x \in \{\alpha, \beta, \gamma\}$.

Now we would like to apply $\mathrm{Hom}_{\mathfrak{b}}(-, L_{\lambda',\mu'})$ to (13) (omitting the term $U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (\mathbb{C} \otimes \mathsf{L}_{\lambda,\mu})$). As components of (13) are free $\mathfrak{c}$-modules of finite rank, the result will be a complex, every component of which is a direct sum of some copies of $L_{\lambda',\mu'}$. To be able to write down the maps explicitly (which will be necessary for our computations), we would need some notation and a rewritten version of (13).

Choose some ordered basis $\mathbf{b} = \{b_1, \ldots, b_k\}$ of $\mathfrak{c}$ consisting of the e_α's for $\alpha \in \Delta_+$, and the h_α's for $\alpha \in \pi \setminus \pi_\lambda$. We can identify

$$
\begin{array}{ccc}
U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (\mathbb{C} \otimes \mathsf{L}_{\lambda,\mu}) & \text{with} & U(\mathfrak{b})/I_{\lambda,\mu}, \\
U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathbb{C} \otimes \mathsf{L}_{\lambda,\mu}) & \text{with} & U(\mathfrak{b}), \\
U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathfrak{c} \otimes \mathsf{L}_{\lambda,\mu}) & \text{with} & \displaystyle\bigoplus_{i=1,\ldots,k} U(\mathfrak{b}), \\
U(\mathfrak{b}) \otimes_{U(\mathfrak{c})} (U(\mathfrak{c}) \otimes \mathfrak{c} \wedge \mathfrak{c} \otimes \mathsf{L}_{\lambda,\mu}) & \text{with} & \displaystyle\bigoplus_{1 \leq i < j \leq k} U(\mathfrak{b}),
\end{array}
$$

such that (13) becomes

$$(14) \qquad \bigoplus_{1 \leq i < j \leq k} U(\mathfrak{b}) \xrightarrow{\gamma_3} \bigoplus_{i=1,\ldots,k} U(\mathfrak{b}) \xrightarrow{\beta_3} U(\mathfrak{b}) \xrightarrow{\alpha_3} U(\mathfrak{b})/I_{\lambda,\mu} \to 0,$$

where α_3 is the natural projection; β_3 is given by

$$B = (\cdot(b_1 - \nu(b_1)), \ldots, \cdot(b_i - \nu(b_i)))$$

(the starting dot means "right multiplication"), and γ_3 is given by the matrix $C = (\cdot c_{ij,s})_{1 \leq i < j \leq k}^{s=1,\ldots,k}$ such that

(15)
$$c_{ij,s} = \begin{cases} b_j - \nu(b_j), & s = i, [b_i, b_j] \neq cb_i \text{ for any } c \in \mathbb{C} \setminus \{0\}; \\ -b_i + \nu(b_i), & s = j, [b_i, b_j] \neq cb_j \text{ for any } c \in \mathbb{C} \setminus \{0\}; \\ b_j - \nu(b_j) + c, & s = i, [b_i, b_j] = cb_i \text{ for some } c \in \mathbb{C} \setminus \{0\}; \\ -b_i + \nu(b_i) - c, & s = j, [b_i, b_j] = cb_j \text{ for some } c \in \mathbb{C} \setminus \{0\}; \\ \frac{1}{c}, & b_s = c[b_i, b_j], c \in \mathbb{C} \setminus \{0\}, s \neq i, j; \\ 0, & \text{otherwise.} \end{cases}$$

Applying $\mathrm{Hom}_{\mathfrak{b}}(\_, L_{\lambda',\mu'})$ to (14) (omitting the non-free term $U(\mathfrak{b})/I_{\lambda,\mu}$), we obtain the following complex:

(16)
$$0 \to L_{\lambda',\mu'} \xrightarrow{\beta_4} \bigoplus_{i=1,\ldots,k} L_{\lambda',\mu'} \xrightarrow{\gamma_4} \bigoplus_{1 \leq i < j \leq k} L_{\lambda',\mu'},$$

where the maps β_4 and γ_4 are given by the matrices B^T and C^T, respectively (and the left instead of the right multiplication with the elements of these matrices).

Now we have to estimate the dimension of the first homology of the complex (16). Split the direct sum $\bigoplus_{i=1,\ldots,k} L_{\lambda',\mu'}$ into two parts: the first one, X, corresponding to $b_i \in [\mathfrak{n}, \mathfrak{n}]$, and the rest, Y. We will need the following lemma:

Lemma 56. *For all $m \geq 0$ the vectorspace $\bigcap_{\alpha \in \pi_{\lambda'}} \mathrm{Ker}(e_\alpha - \lambda'(\alpha))^m$ of $L_{\lambda',\mu'}$ is finite dimensional.*

Proof. In the case $|\pi_{\lambda'}| = 1$ this follows from the description of the action of $e_\alpha - \lambda'(\alpha)$ in Subsection 6.1. As the general case is a tensor power of the case considered in Subsection 6.1 (see (11)), the general claim follows from the fact that the tensor product of finite dimensional spaces is finite dimensional. $\square$

As $[\mathfrak{n}, \mathfrak{n}]L_{\lambda',\mu'} = 0$, the image of β_4 belongs to Y. Let now $x = (x_i) \in \bigoplus_{i=1,\ldots,k} L_{\lambda',\mu'}$ and assume that $x \notin Y$. Then $x_i \neq 0$ for some component i from X. Assume for the moment that b_i is central in $\mathfrak{n}$. Let $b_j \in \mathfrak{b}$, $j \neq i$. Then from (15) one obtains that the ij-th (or the ji-th, depending on the ordering of i and j) component of $\gamma_4(x)$ equals either $\pm(b_j - \nu(b_j))x_i$ or $\pm(b_j - \nu(b_j) + c)x_i$. Hence $\gamma_4(x) = 0$ implies that x_i must be a Whittaker vector in $L_{\lambda',\mu'}$ (note that the subspace of all Whittaker vectors in $L_{\lambda',\mu'}$ is one-dimensional). If b_i is such that $[b_i, \mathfrak{n}]$ is in the center of $\mathfrak{n}$, then we can apply similar arguments and get (from (15)) that either $\pm(b_j - \nu(b_j))x_i = 0$ (resp.

$\pm(b_j - \nu(b_j) + c)x_i = 0$) or $\pm(b_j - \nu(b_j))x_i$ (resp. $\pm(b_j - \nu(b_j) + c)x_i$) is proportional to the component of $[b_i, b_j]$, in which case both $b_i, b_j \in \mathfrak{n}$. The latter means that $[b_i, b_j]$ is central in $\mathfrak{n}$ and thus $\pm(b_j - \nu(b_j))x_i$ (resp. $\pm(b_j - \nu(b_j) + c)x_i$) is proportional to the Whittaker vector in $L_{\lambda',\mu'}$. This means that x_i must be in the subspace $\bigcap_{\alpha \in \pi_{\lambda'}} \mathrm{Ker}(e_\alpha - \lambda'(\alpha))^2$ of $L_{\lambda',\mu'}$, which is finite dimensional by Lemma 56. Proceeding by induction, we obtain that every x_i for a component from X belongs to some fixed subspace of the form $\bigcap_{\alpha \in \pi_{\lambda'}} \mathrm{Ker}(e_\alpha - \lambda'(\alpha))^m$, which is finite dimensional by Lemma 56.

The above means that $\mathrm{Ker}(\gamma_4)/(\mathrm{Ker}(\gamma_4) \cap Y)$ is finite dimensional. At the same time, the first homology of (16) coming from Y corresponds exactly to $\mathrm{Ext}^1_{\mathfrak{a}}(L_{\lambda,\mu}, L_{\lambda',\mu'})$ by the same construction as in the first part of the proof, applied to the algebra $\mathfrak{a}$. This part is finite dimensional by Proposition 54. It follows that the whole first homology of (16) is finite dimensional, which completes the proof. $\qquad\square$

The natural inclusion $\mathfrak{h}_\lambda \hookrightarrow \mathfrak{h}$ induces the natural projection $\mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*_\lambda$. Let G denote the image of the abelian group $\mathbb{Z}\Delta$ under this projection. For $\xi \in \mathfrak{h}^*_\lambda/G$ denote by $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi$ the Serre subcategory of $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}$, generated by $L_{\lambda,\mu}$, $\mu \in \xi$. Using the usual theory of weight and generalized weight modules one easily proves the following block decomposition for the category $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}$:

$$\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)} = \bigoplus_{\xi \in \mathfrak{h}^*_\lambda/G} \overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi.$$

An interesting question is to describe $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi$ via quiver and relations similarly to Theorem 52(ii). Note that by Proposition 1 we can use Corollary 55 to compute extensions in $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi$. By this Corollary, the ext-quiver of $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi$ is locally finite. Motivated by the results from Subsection 6.1 it is natural to expect that the algebra describing $\overline{\mathfrak{W}^{\mathfrak{b}}_{\mathfrak{n}}(\lambda)}_\xi$ is Koszul. From the proof of Corollary 55 it is easy to see this algebra is more complicated than the tensor products of algebras from Subsection 6.1 (such tensor products are obviously Koszul). If one makes a parallel with simple finite dimensional Lie algebras, then our expectation of Koszulity for this algebra is similar to Alexandru conjecture (for thick category $\mathcal{O}$), see [Gai].

6.4. Some solvable subquotients of the Virasoro algebra.
Another way to generalize the results of Subsection 6.1 is to consider certain subquotients of the Virasoro algebra. For $n = 0, 1, 2, \dots$ let $\mathfrak{v}_n$ denote the Lie algebra with the basis $\{e_i : i = n, n+1, \dots\}$, and the Lie bracket given by $[e_i, e_j] = (j - i)e_{i+j}$. For $n > 0$ the algebra $\mathfrak{v}_n$ is quasi-nilpotent. For $k \geq n$ the algebra $\mathfrak{v}_k$ is an ideal of $\mathfrak{v}_n$. The

quotient $\mathfrak{v}_n/\mathfrak{v}_k$ is always solvable and, moreover, nilpotent if $n > 0$. In particular, it is easy to see that $(\mathfrak{v}_n, \mathfrak{v}_k)$ is a Whittaker pair for all $k > n$ and $(\mathfrak{v}_n/\mathfrak{v}_k, \mathfrak{v}_m/\mathfrak{v}_k)$ is a Whittaker pair for all $k > m > n$. For $n = 0$, $m = 1$ and $k = 2$ one obtains the algebra considered in Subsection 6.1. For $n = 1$, $m = 3$ and $k = 4$ one obtains the algebra considered in Subsection 6.2.

For all Whittaker pairs $(\mathfrak{g}, \mathfrak{n})$ of the form $(\mathfrak{v}_0, \mathfrak{v}_1)$ and $(\mathfrak{v}_0/\mathfrak{v}_k, \mathfrak{v}_1/\mathfrak{v}_k)$ the module M_λ is isomorphic to $\mathbb{C}[e_0]$ as a $\mathbb{C}[e_0]$-module. It is simple if and only if $\lambda \neq 0$ (the "if" part follows from Proposition 50 and the "only if" part is obvious). If $\lambda = 0$, then M_λ is free of rank one over its endomorphism algebra $\mathrm{End}_{U(\mathfrak{g})}(M_\lambda) \cong \mathbb{C}[e_0]$ and simple quotients of M_λ are all one-dimensional and have the form L_μ, where $\mu \in \mathbb{C}$, $\mathfrak{n}L_\mu = 0$ and $e_0 v = \mu v$ for all $v \in L_\mu$. Similarly to Corollary 55 one can show that indecomposable blocks of the category $\overline{\mathfrak{W}_{\mathfrak{n}}^{\mathfrak{b}}}$ can be described as module categories over (completions of) some locally finite quiver algebras with relations. We believe that these algebras are Koszul.

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